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Le Mans Université • CNRS - UMR 6613

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Graduate School
Le Mans Université

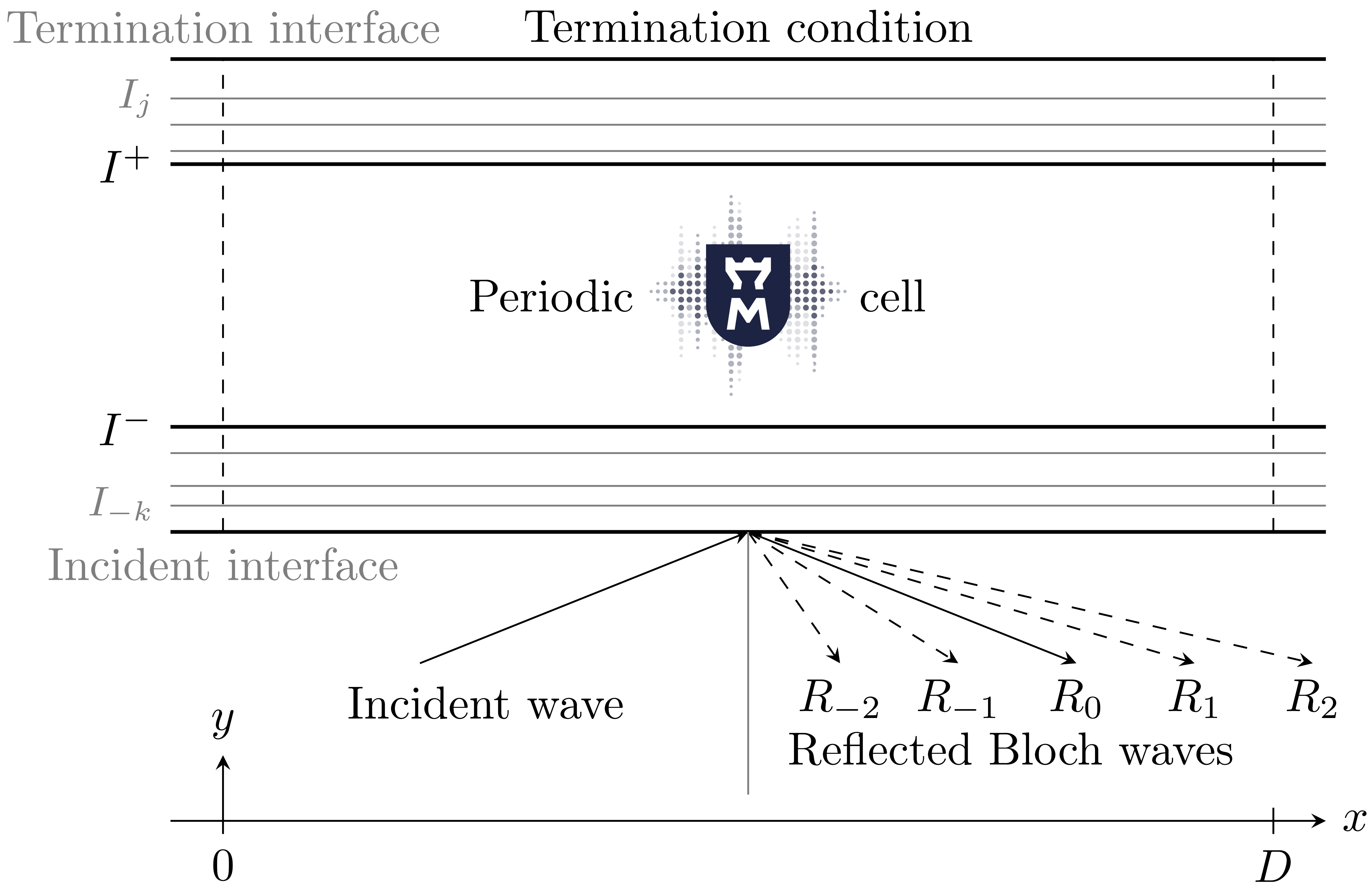
Sorrento **SAPEM' 23** 常熟
Montabon F-72500

A recursive method to evaluate the scattering properties of grating stacks

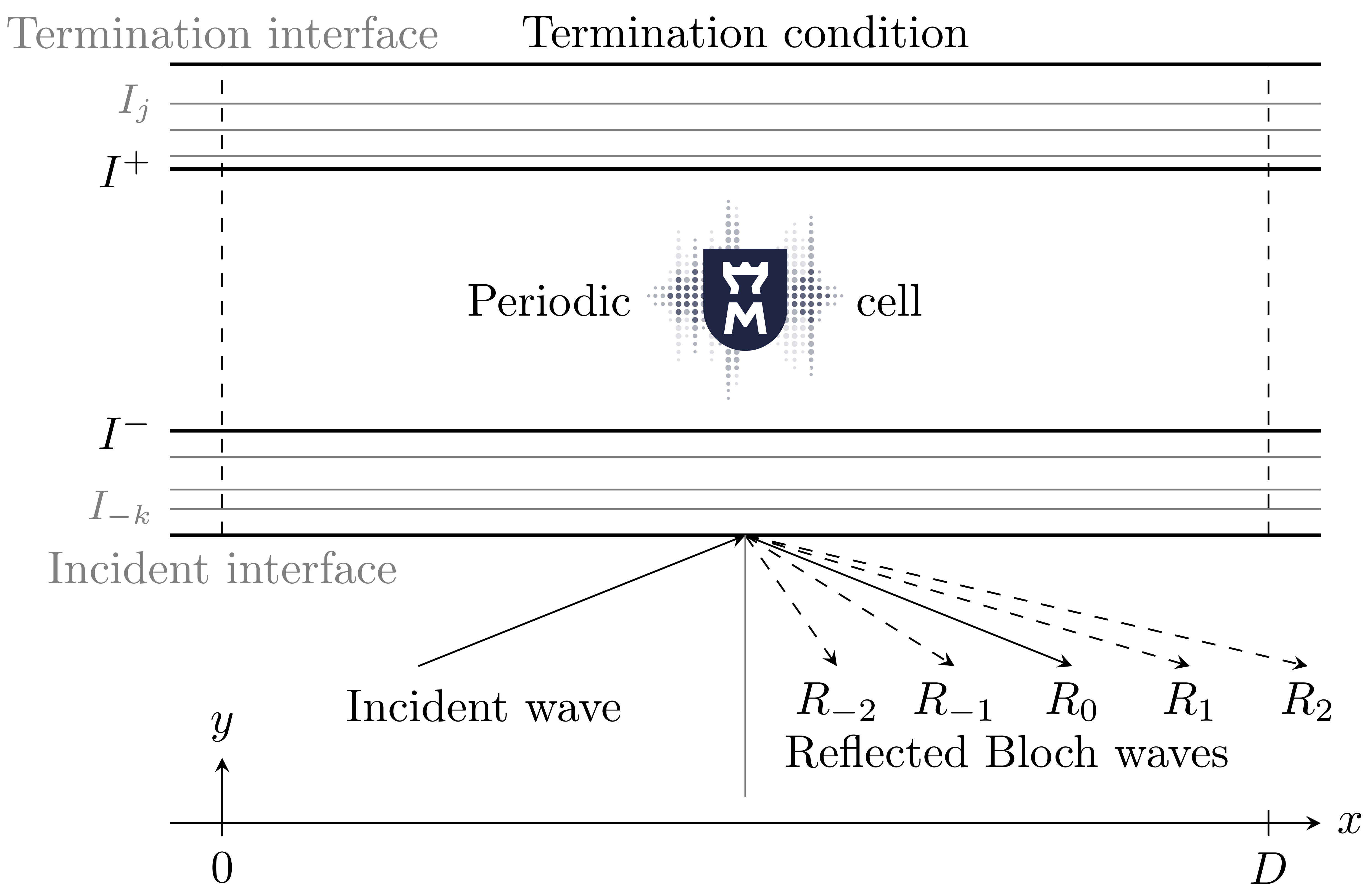
O. Dazel, M. Gaborit, and J.-P. Groby

Laboratoire d'Acoustique de l'Université du Mans (LAUM - UMR CNRS 6613)
Institut d'Acoustique - Graduate School (IA-GS) | Le Mans Université, France

Models for the acoustic response of grating stacks



Models for the acoustic response of grating stacks



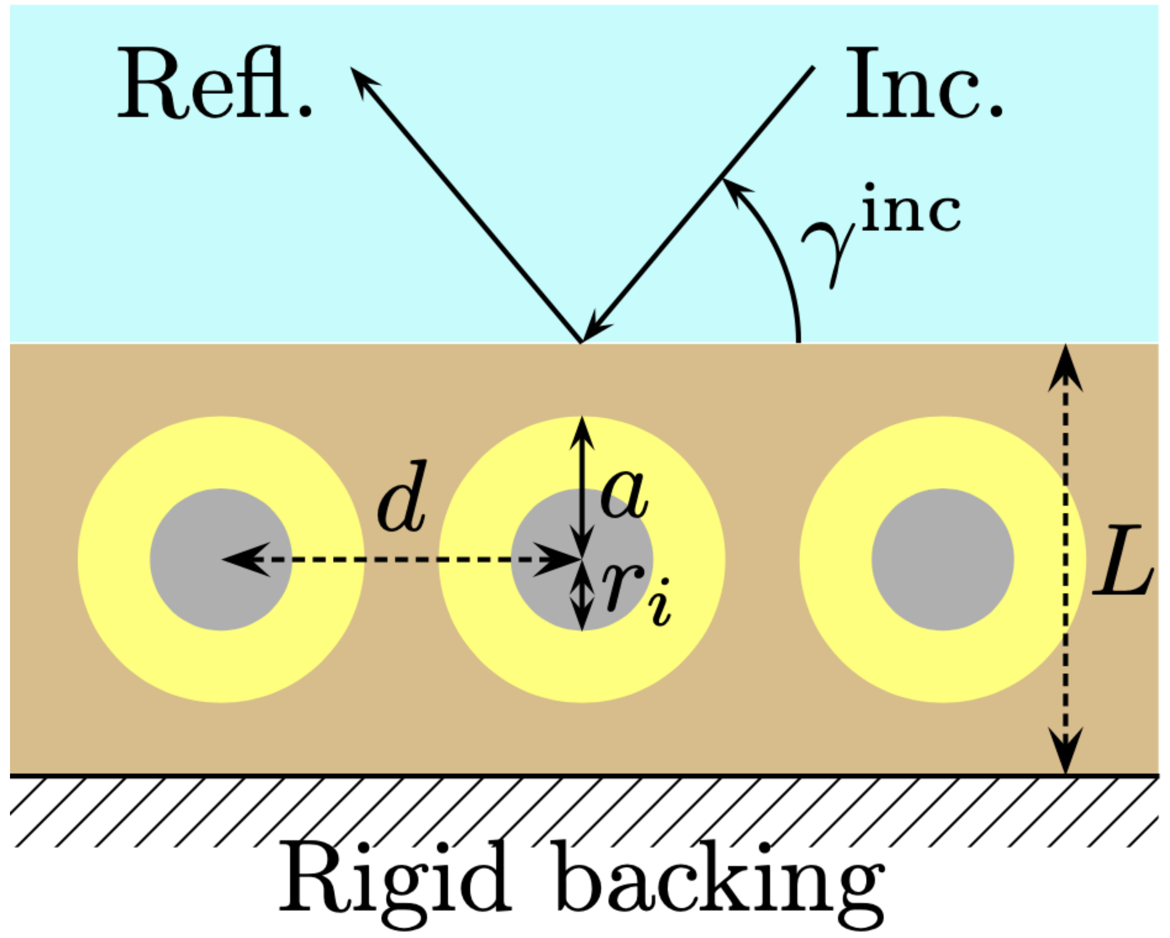
Comsol
Not the exact solution, Lots of dofs, PML, not fun …

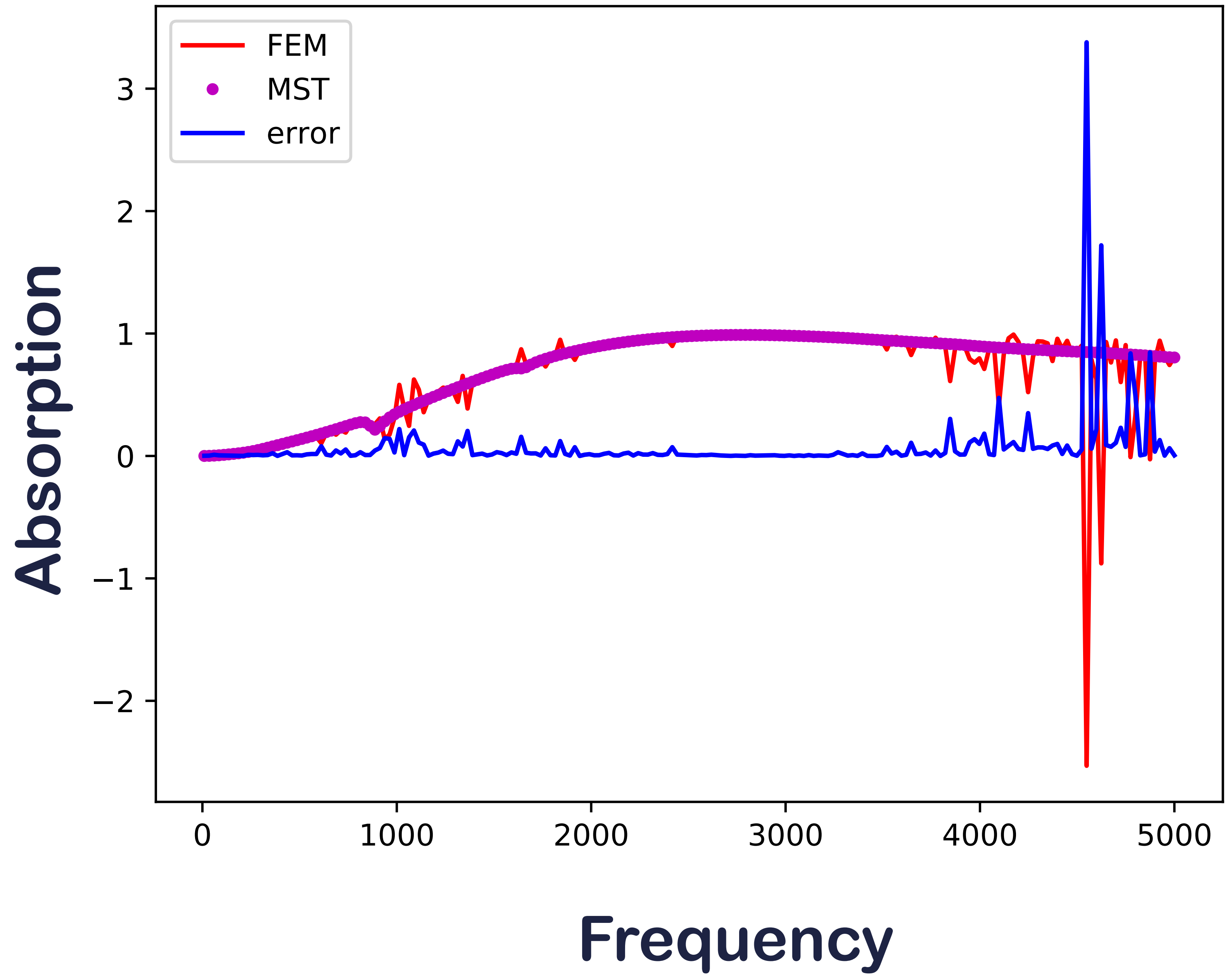
Gaborit et al AAA (2018)
Case of elastic plate, coupled problem

Parinello et al. JSV (2019)
Periodic TMM,
only nodal dofs (limited to quadratic elements)

Dazel et al. (CFA 2022)
Generalisation of Parinello’s method,
available for higher orders

TMM and evanescent waves : stability issues





- Poroelastic core
- Bicomposite poroelastic steel inclusion

Origin of the instabiities

- Evanescent waves in TMM
- Boundary conditions in FEM

Objective of the presentation:
Adapt a stable recursive scheme to grating stacks (couple it with periodic FEM)

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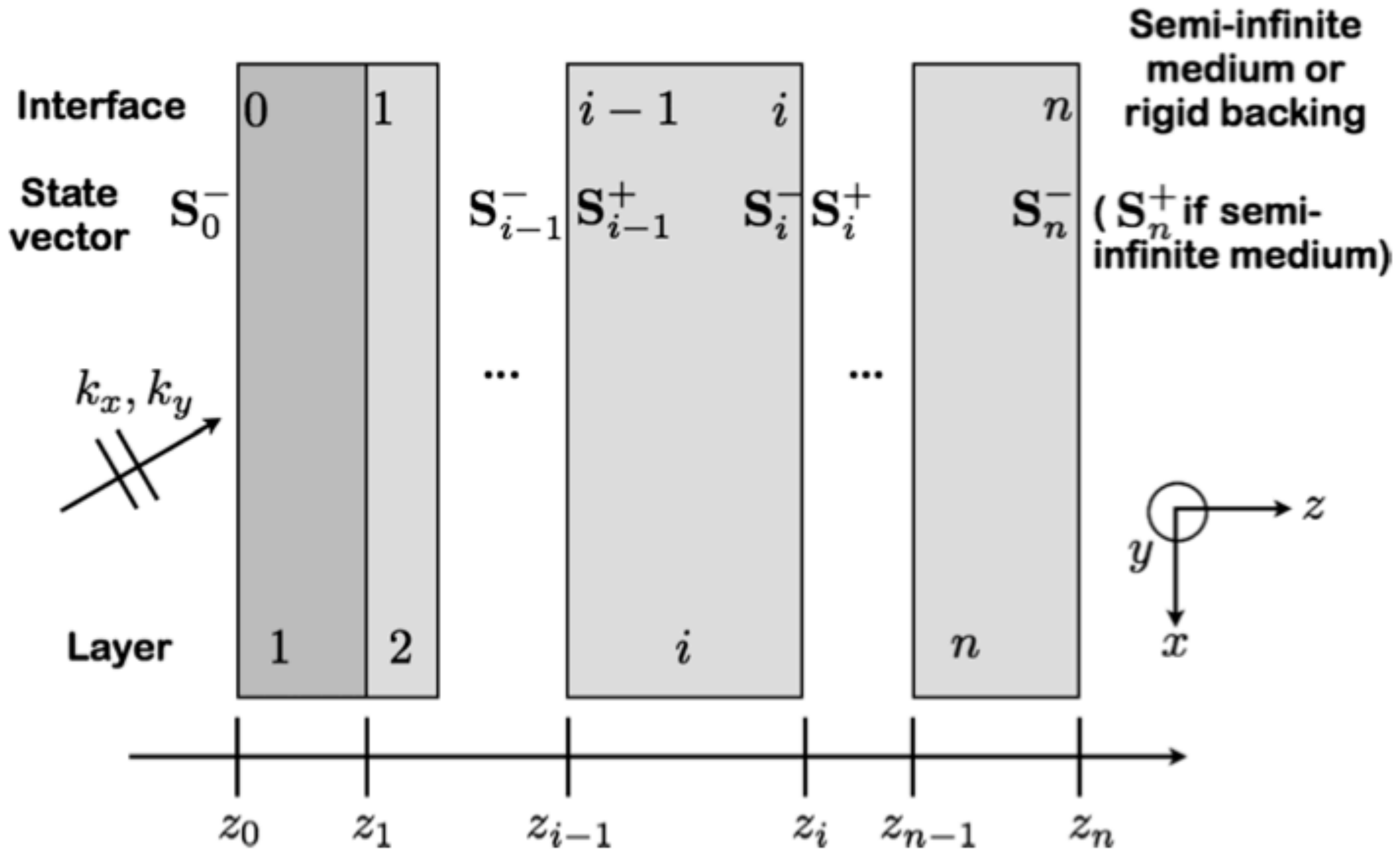
The Transfer Matrix Method

Thomson, JAP (1950)
Haskell, Bull. Seismol Soc. (1953)
Brouard et al., JSV (1995)



- **Multilayer (plane) structures**
- **Plane (or quasi-plane if anisotropy) waves**
- **“Cheap”**

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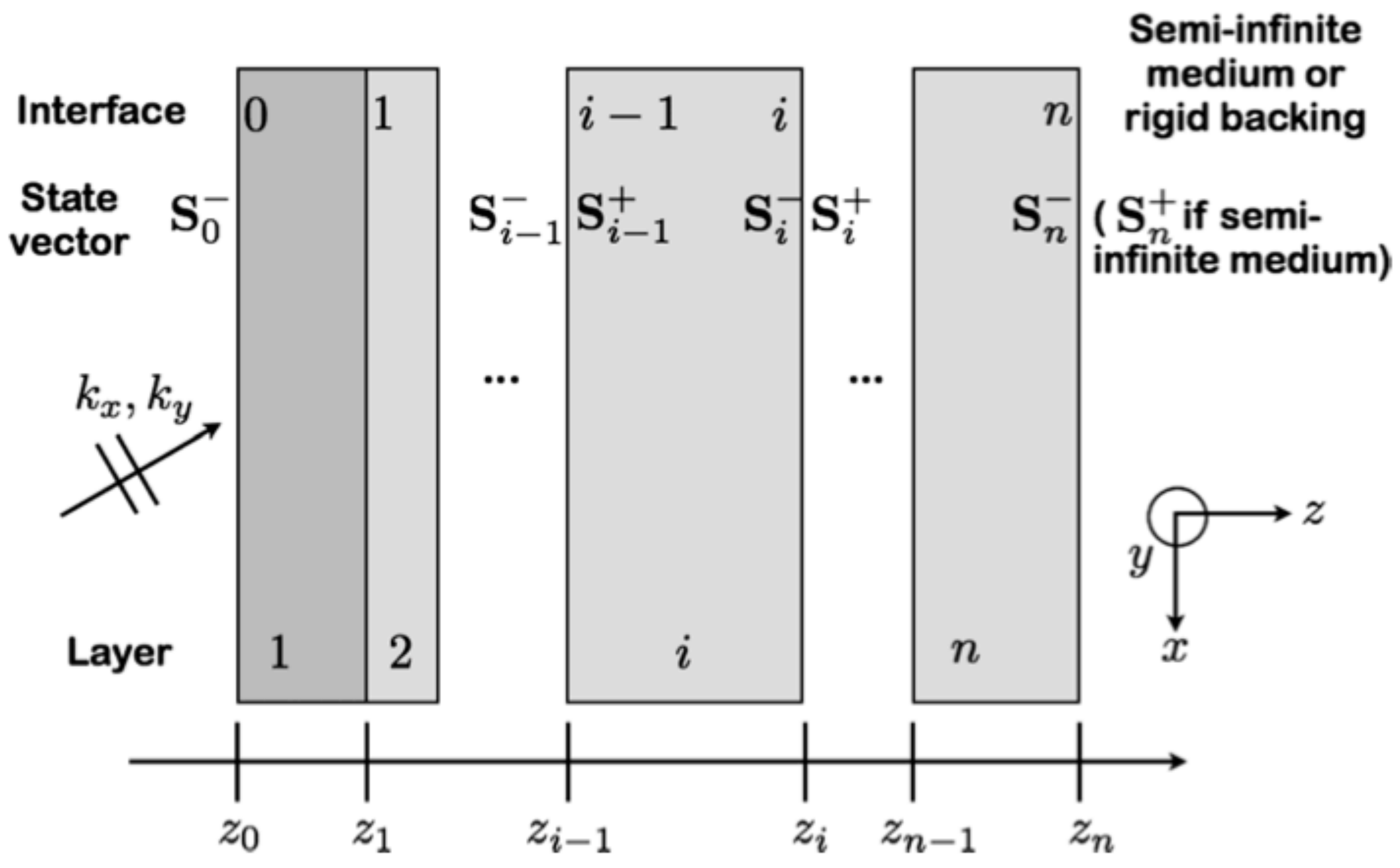

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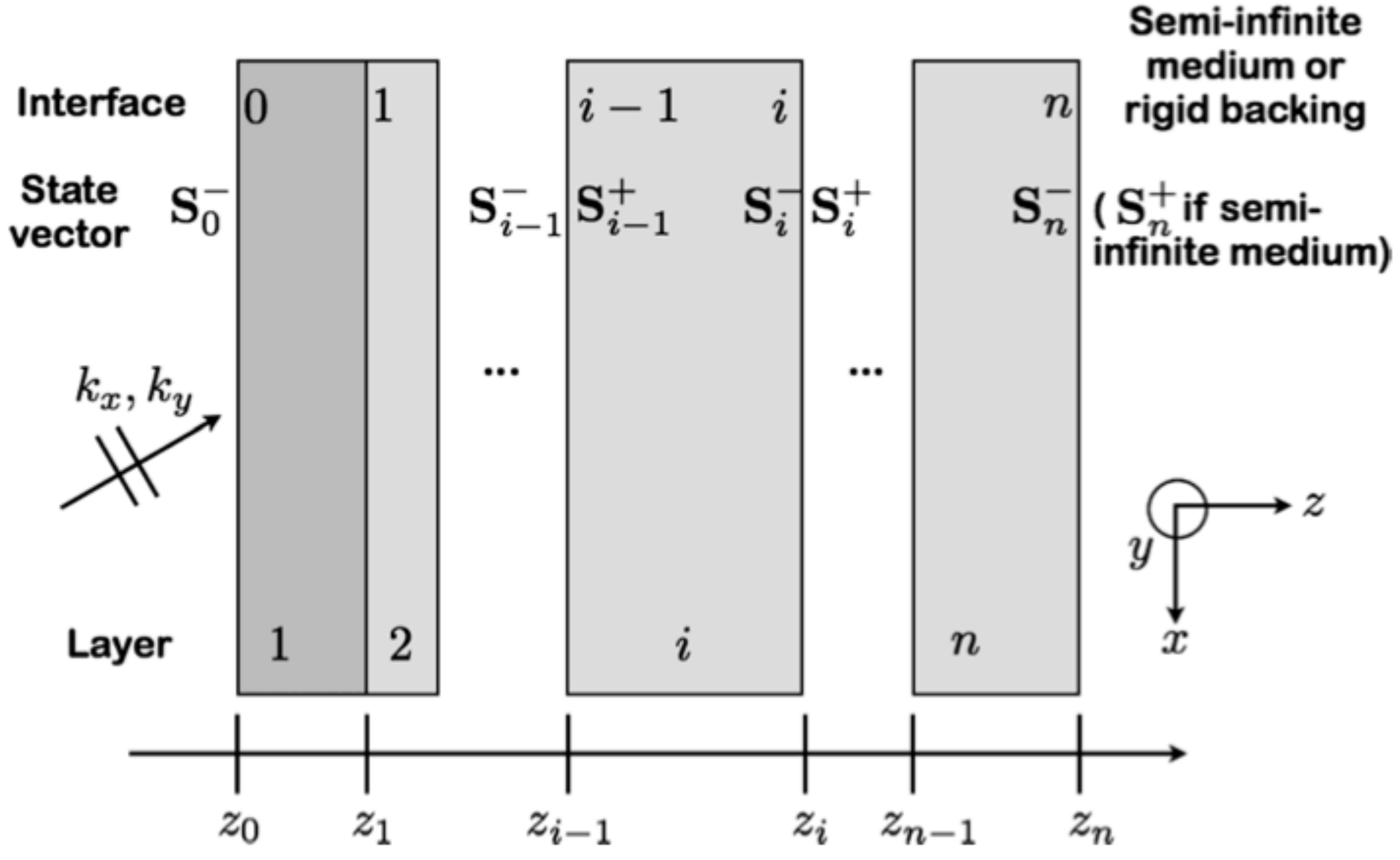

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- Choice of physical fields in the State Vector
- Relations at interfaces
  - Continuity of displacements
  - Continuity of stress
- Unstable
  - By nature
  - High frequencies
  - Highly damped materials

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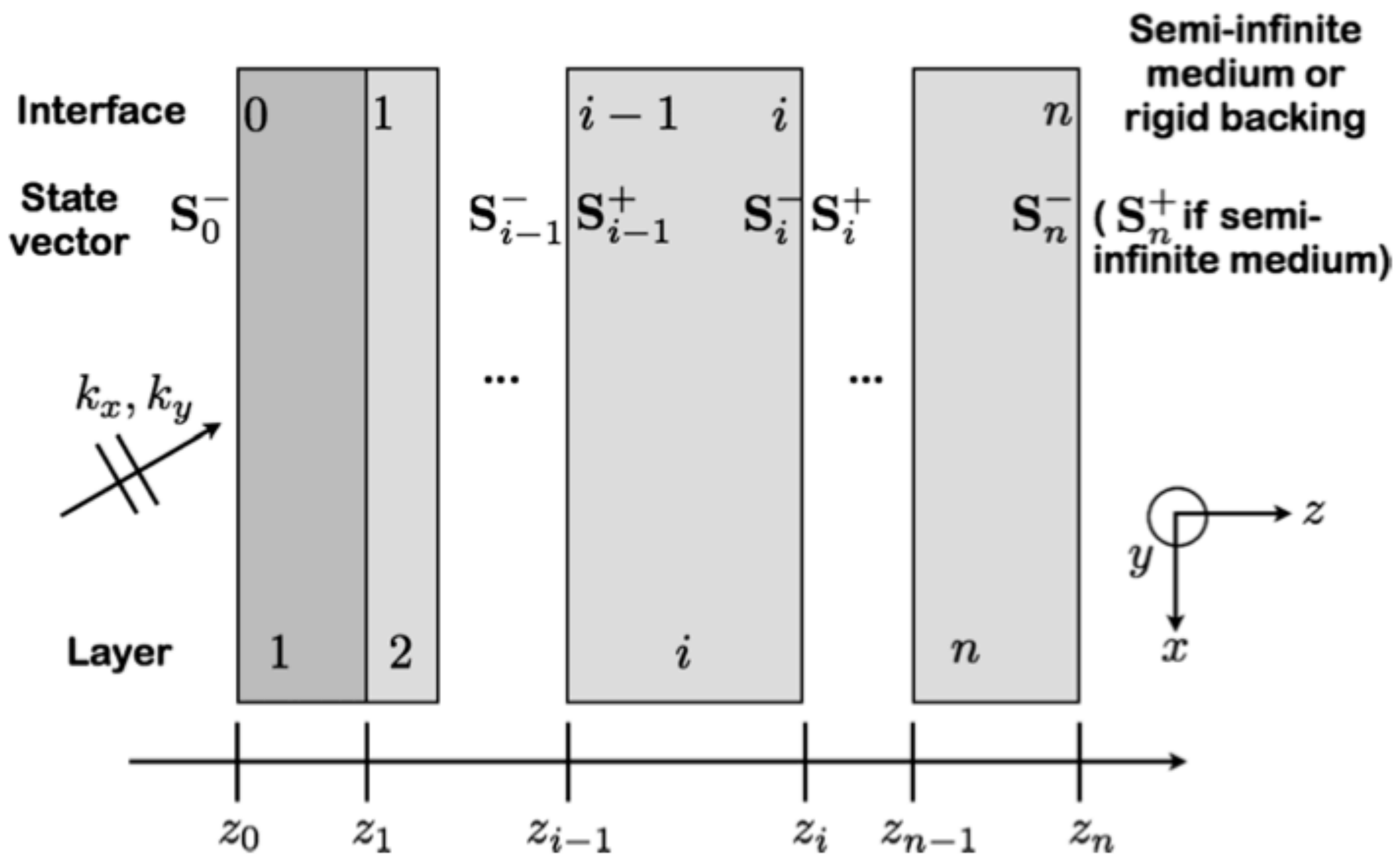

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Medium of same type

$$[\mathbf{T}] = \prod_{i=1}^{n} [\mathbf{T}_i]$$

If not: a global system

- Choice of physical fields in the State Vector
- Relations at interfaces
  - Continuity of displacements
  - Continuity of stress
- Unstable
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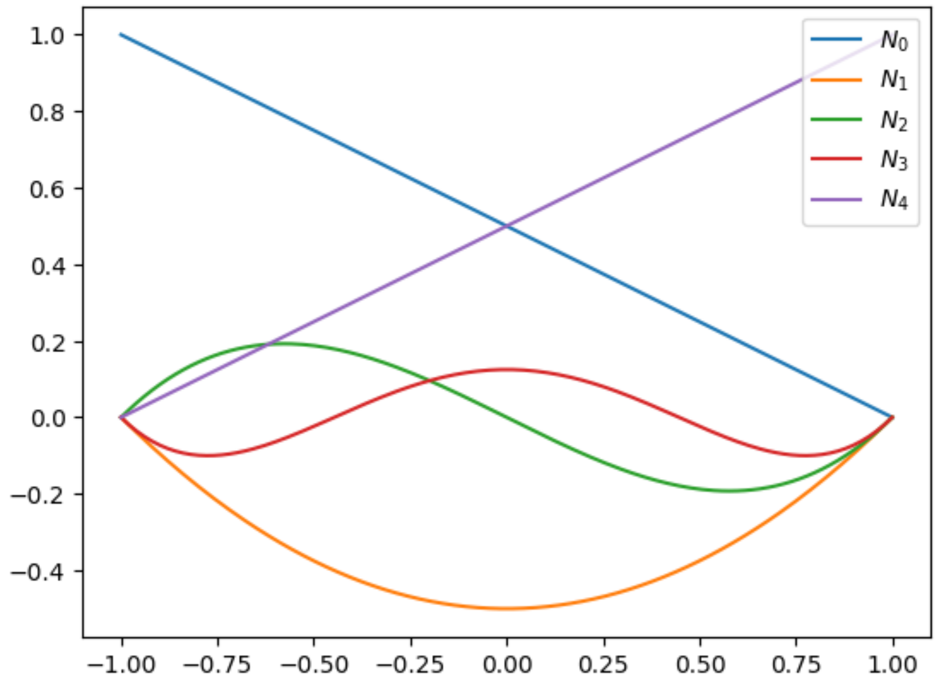
Finite-Element Method (higher order)

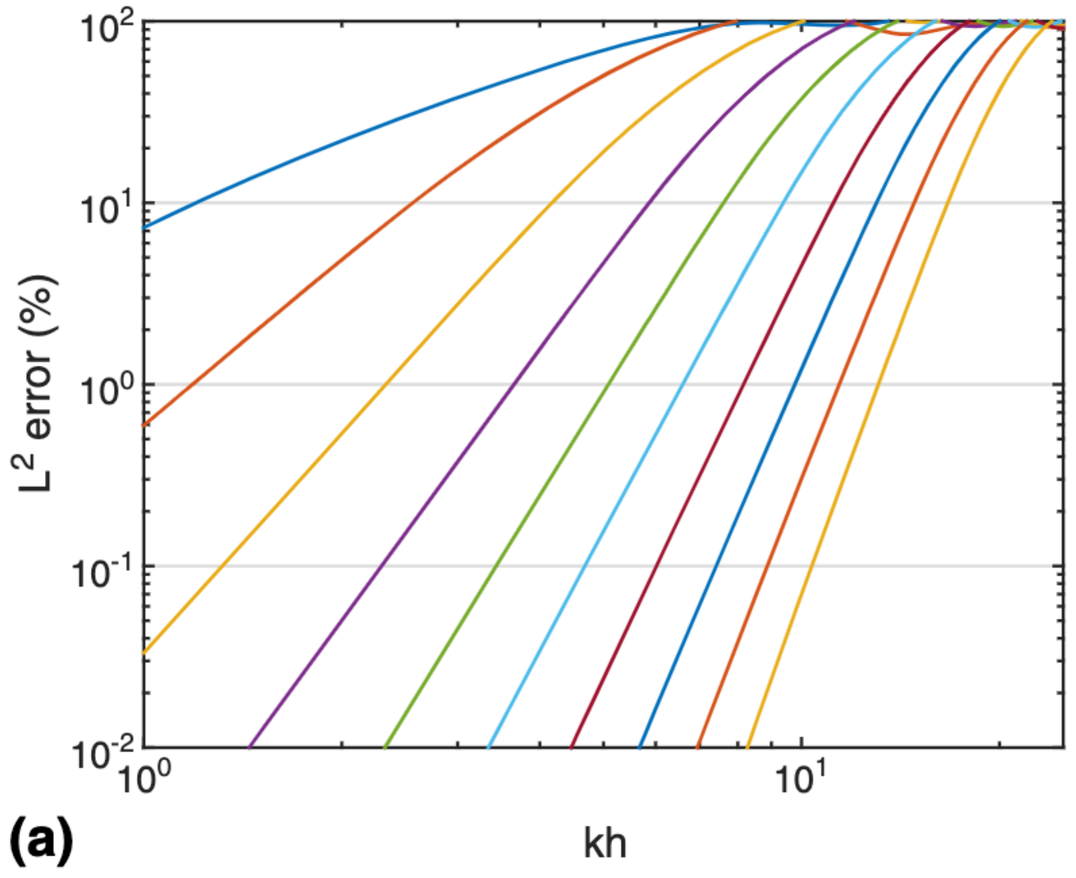
Weak form for a fluid

Šolín et al. Chapman & Hall (2003)
Bériot et al. IJNME (2016)
Jonckheere et al. IJNME (2022)

$$\forall q, \quad \int_\Omega \boldsymbol{\nabla} p \; \boldsymbol{\nabla} q - k^2 pq \,\mathrm{d}\Omega = \int_{\partial\Omega} \frac{\partial p}{\partial n} q \,\mathrm{d}S$$

Lobatto shape functions





Stabilised implementation

$$\forall q, \quad \int_\Omega \boldsymbol{\nabla} p \; \boldsymbol{\nabla} q - k^2 pq \,\mathrm{d}\Omega + \int_{\partial\Omega} ikpq \,\mathrm{d}S = \int_{\partial\Omega} \left(\frac{\partial p}{\partial n} + ikp \right) q \,\mathrm{d}S$$

Incoming characteristics

Concept of characteristics

$$\left[\boldsymbol{P}_1^+ \quad | \quad \boldsymbol{P}_1^-\right]\begin{Bmatrix} q_1^+ \\ q_1^- \end{Bmatrix} = \boldsymbol{S}_1 = \begin{Bmatrix} \dfrac{\partial p_1}{\partial n} \\ p_1 \end{Bmatrix}$$

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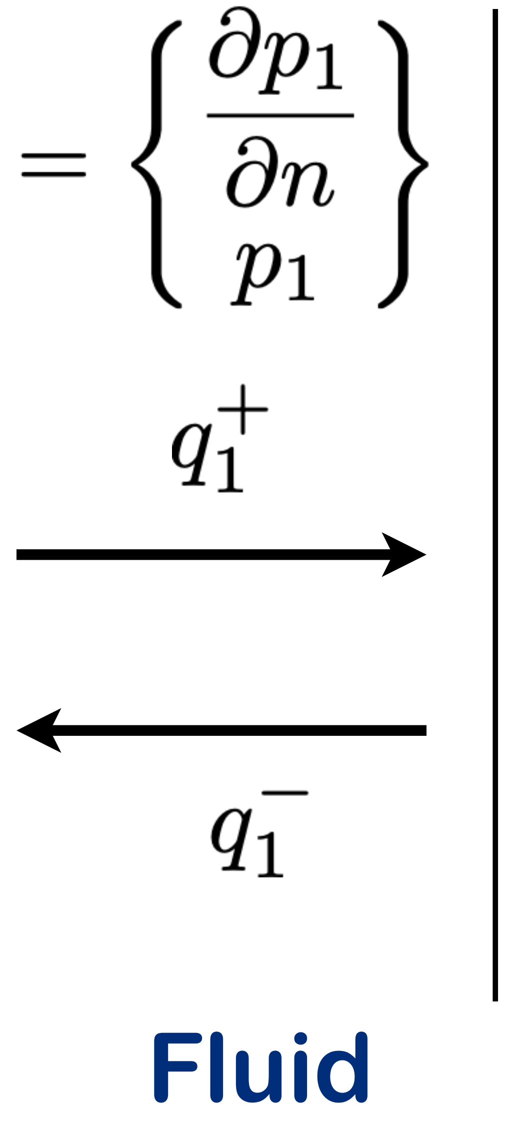

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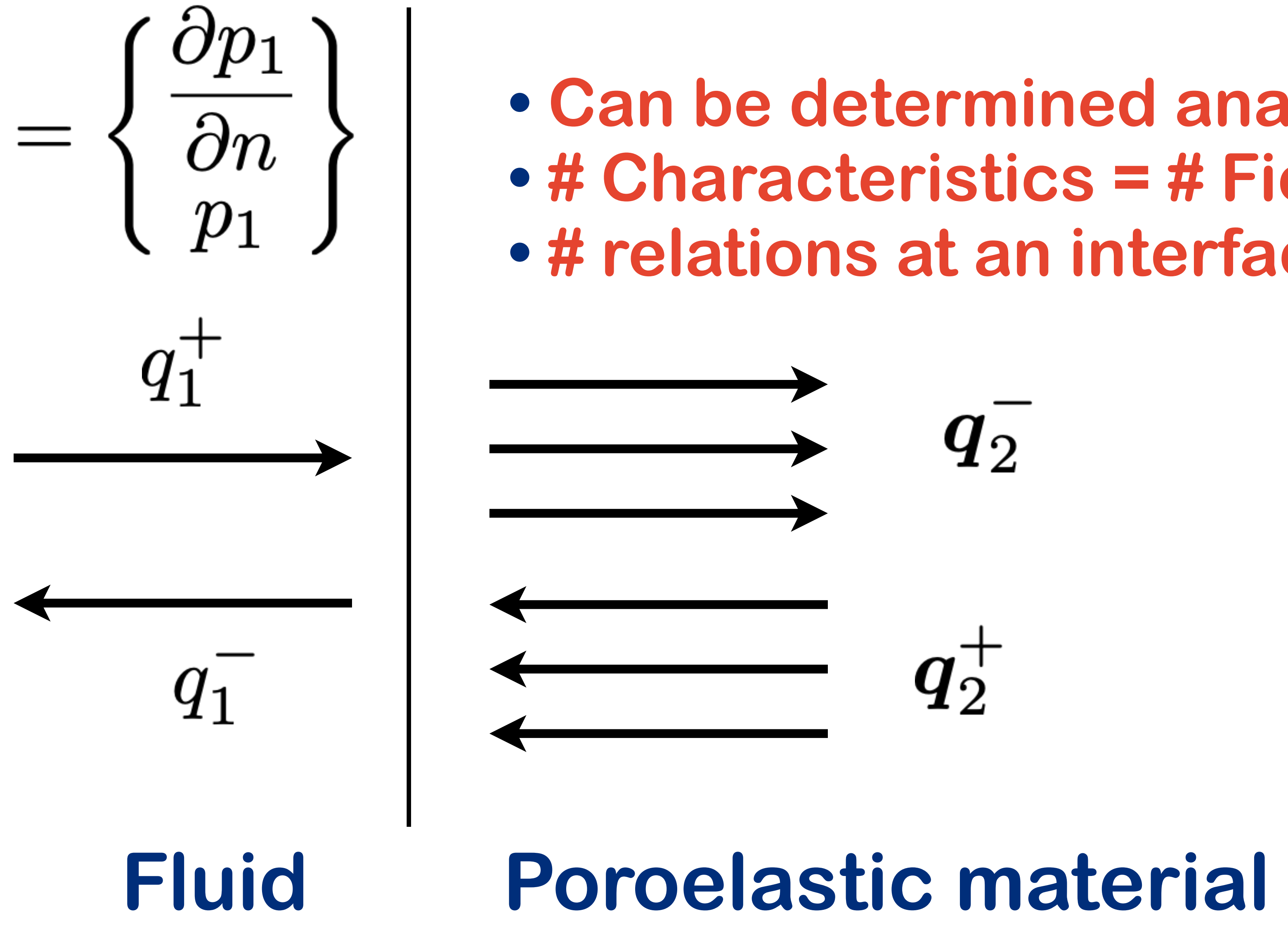
- Can be determined analytically (for all types of medium)
- # Characteristics = # Fields in the SV
- # relations at an interface = # incoming characteristics



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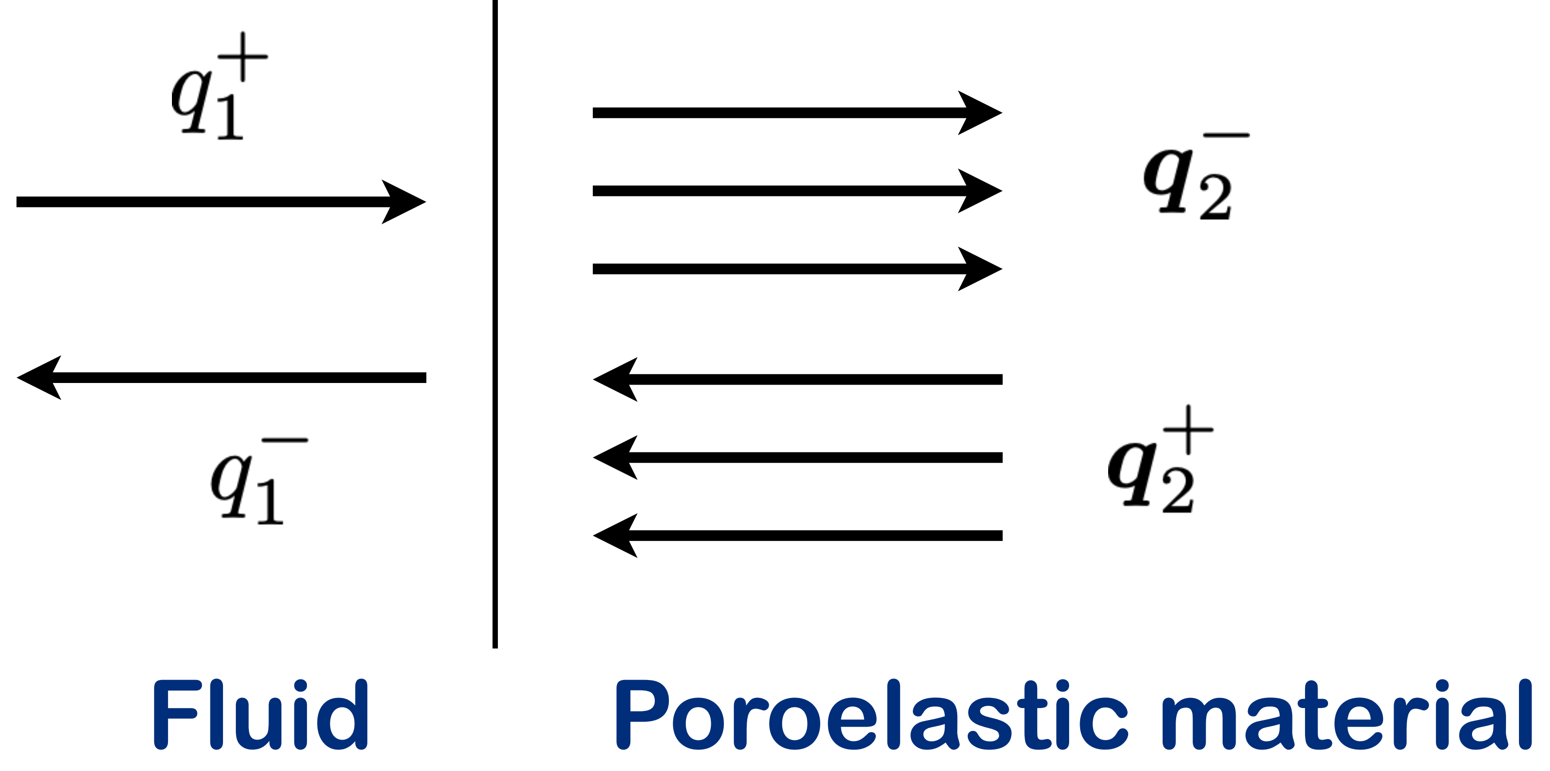


Well-posedness problem [Hadamard; Kreiss, Higdon]:

$$\begin{Bmatrix} q_1^- \\ \boldsymbol{q}_2^+ \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{R}_{11}] & [\boldsymbol{T}_{12}] \\ [\boldsymbol{T}_{12}] & [\boldsymbol{R}_{22}] \end{bmatrix} \begin{Bmatrix} q_1^+ \\ \boldsymbol{q}_2^- \end{Bmatrix}$$

- Discontinuous Galerkin Methods
- Mode matching
- Domain decomposition
- No a priori choice

Recursive method

- Fix the stability problem of TMM
- Redaction could have been better

Dazel et al. JAP 2013

$$\mathbf{S}_i^{\pm} = [\mathbf{\Omega}_i^{\pm}]\mathbf{X}_i^{\pm}$$

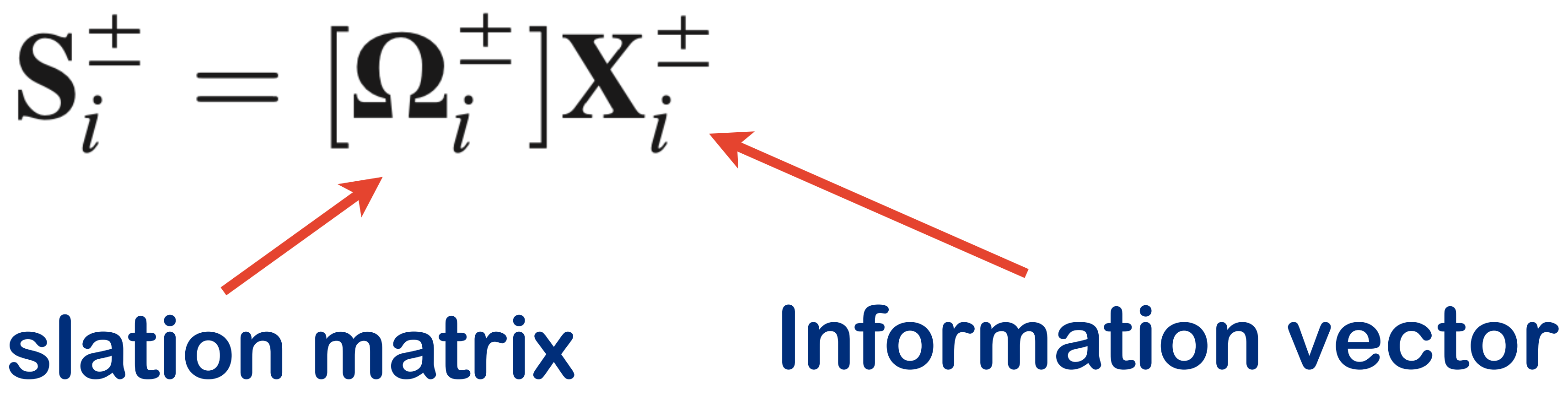
Translation matrix

Information vector

Recursive method

- **Fix the stability problem of TMM**
- **Redaction could have been better**

Dazel et al. JAP 2013



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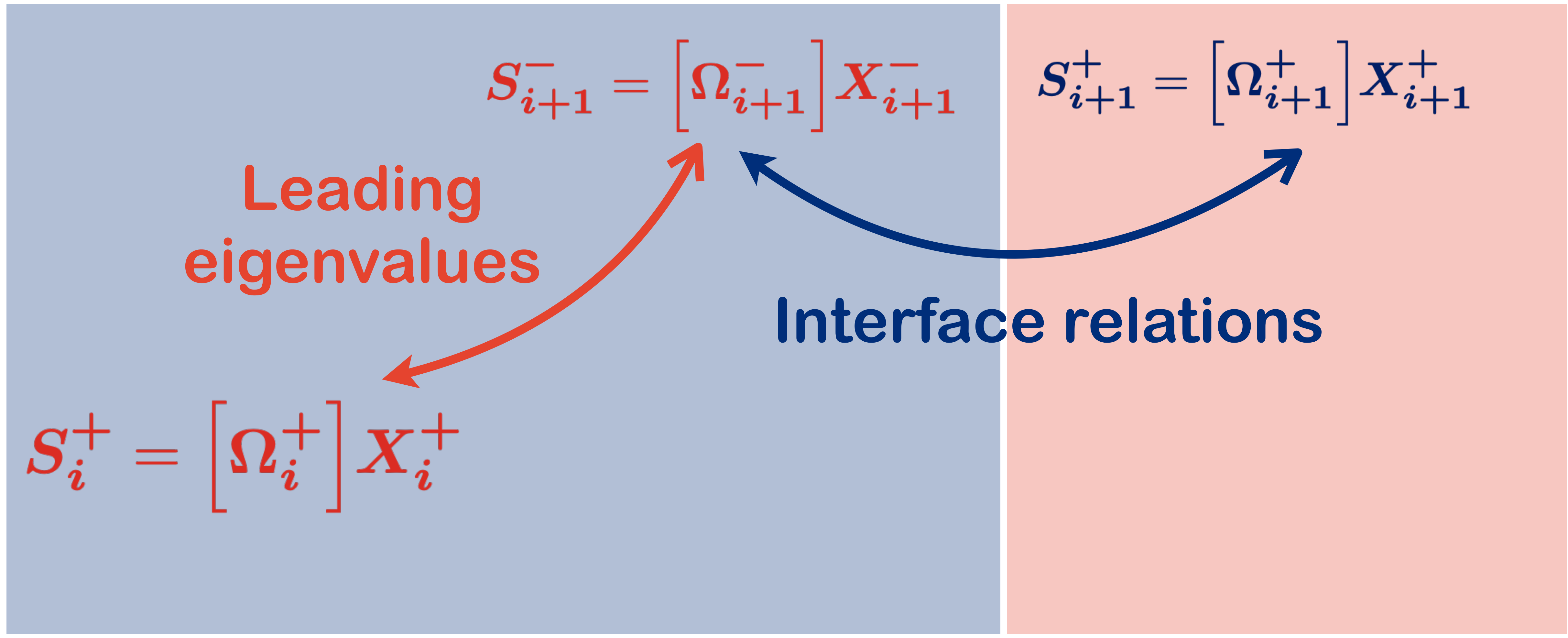

$$\boldsymbol{S}_{i+1}^{-} = \left[\boldsymbol{\Omega}_{i+1}^{-}\right]\boldsymbol{X}_{i+1}^{-}$$

$$\boldsymbol{S}_{i+1}^{+} = \left[\boldsymbol{\Omega}_{i+1}^{+}\right]\boldsymbol{X}_{i+1}^{+}$$

Leading eigenvalues

Interface relations

$$\boldsymbol{S}_{i}^{+} = \left[\boldsymbol{\Omega}_{i}^{+}\right]\boldsymbol{X}_{i}^{+}$$

```python
_list = [0.]*(m-1)+[1.] +[np.exp(-(lambda_[m+i]-lambda_[m-1])*self.d) for i in range(m)]
Lambda = np.diag(np.array(_list))
alpha_prime = Phi.dot(Lambda).dot(Phi_inv) # Eq (21)

xi_prime = Phi_inv[:m,:] @ Om # Eq (23)
_list = [np.exp(-(lambda_[m-1]-lambda_[i])*self.d) for i in range(m-1)] + [1.]
xi_prime_lambda = LA.inv(xi_prime).dot(np.diag(_list))
Om = alpha_prime.dot(Om).dot(xi_prime_lambda)

Om[:,:m-1]  += Phi[:,:m-1]

Xi = xi_prime_lambda*np.exp(lambda_[m-1]*self.d)
return Om, Xi
```

Recursive method

- Fix the stability problem of TMM
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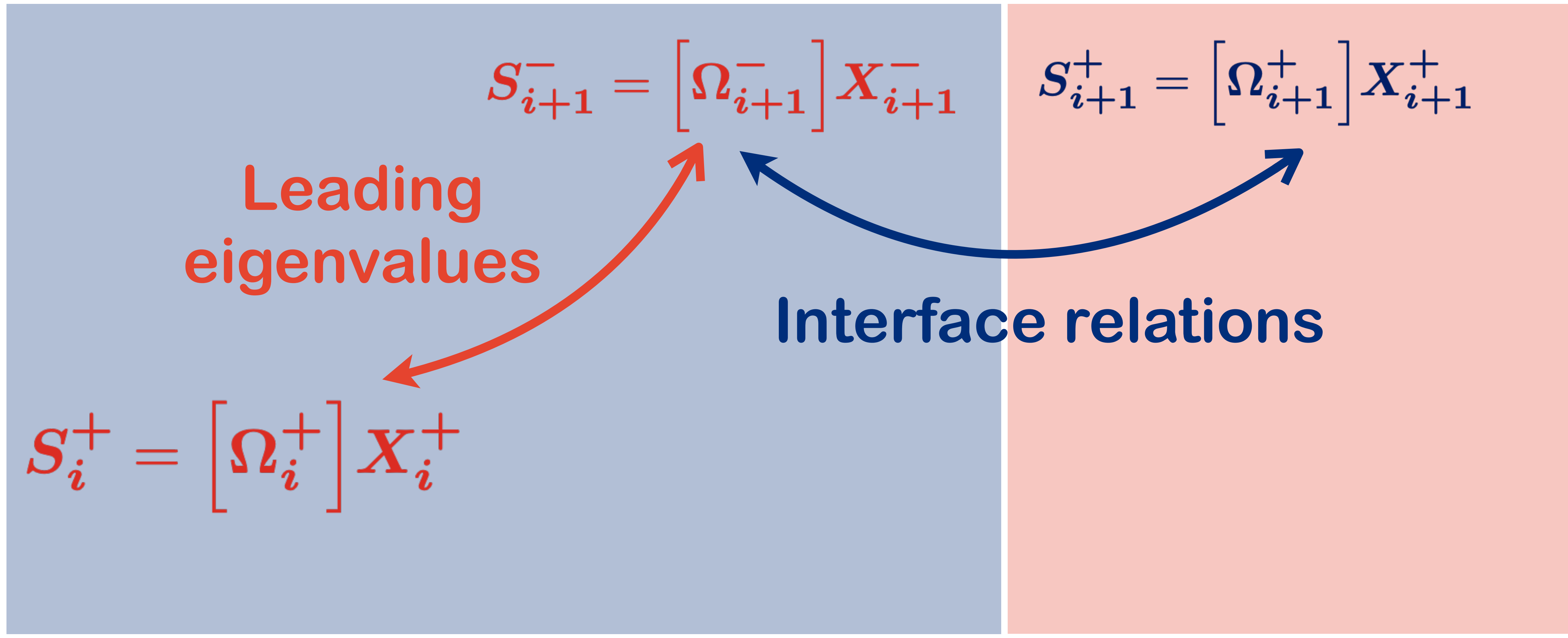
$$\mathbf{S}_i^{\pm} = [\mathbf{\Omega}_i^{\pm}]\mathbf{X}_i^{\pm}$$

Translation matrix **Information vector**



$$S_{i+1}^{-} = \left[\Omega_{i+1}^{-}\right] X_{i+1}^{-}$$

$$S_{i+1}^{+} = \left[\Omega_{i+1}^{+}\right] X_{i+1}^{+}$$

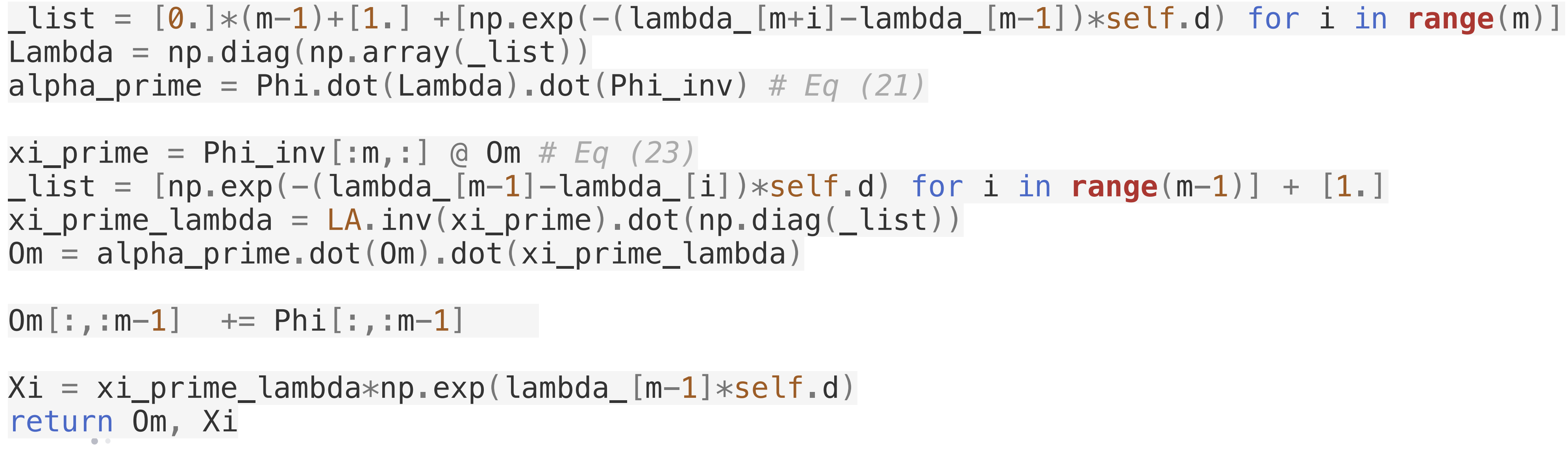
Leading eigenvalues

Interface relations

$$S_i^{+} = \left[\Omega_i^{+}\right] X_i^{+}$$

- All types of physical media
- Extension to anisotropic materials Parra-Martinez et al. JAP 2016
- Code available on github https://github.com/cpplanes/pymls
- No global system

- Significance of the information vector ?
- A-priori in the interface relations
- Not valid at normal incidence



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_list = [np.exp(-(lambda_[m-1]-lambda_[i])*self.d) for i in range(m-1)] + [1.]
xi_prime_lambda = LA.inv(xi_prime).dot(np.diag(_list))
Om = alpha_prime.dot(Om).dot(xi_prime_lambda)

Om[:,:m-1]  += Phi[:,:m-1]

Xi = xi_prime_lambda*np.exp(lambda_[m-1]*self.d)
return Om, Xi
```

See also
Song et al. JSV 2023

Adaptation of the recursive scheme

- Characteristics stabilise FEM predictions
- Significance of the information vector ?
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$$\boldsymbol{\Sigma}_i^+ = \begin{Bmatrix} \boldsymbol{q}_i^+ \\ \boldsymbol{q}_i^- \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{I}_{n_i}] \\ [\boldsymbol{R}_{n_i}] \end{bmatrix} \boldsymbol{q}_i^+$$

- Method can be extended straighforwardly
- Just a change of variables

$$\boldsymbol{S}'(z) = [\boldsymbol{\alpha}]\boldsymbol{S}(z) \qquad \boldsymbol{S}_i^- = [\boldsymbol{T}]\boldsymbol{S}_{i-1}^+$$

$$\boldsymbol{\Sigma}'(z) = [\boldsymbol{\alpha_c}]\boldsymbol{\Sigma}(z) \qquad \boldsymbol{\Sigma}_i^- = [\boldsymbol{T_c}]\boldsymbol{\Sigma}_{i-1}^+$$

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- Information vector = characteristics
- No a-priori in the interface relations
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- Reformulation of the TMM
- No interest if only homogeneous layers
- Transfer/State matrices diagonal at normal incidence
- Wave splitting (inhomogeneous materials)
- Childhood memories

- Information vector = characteristics
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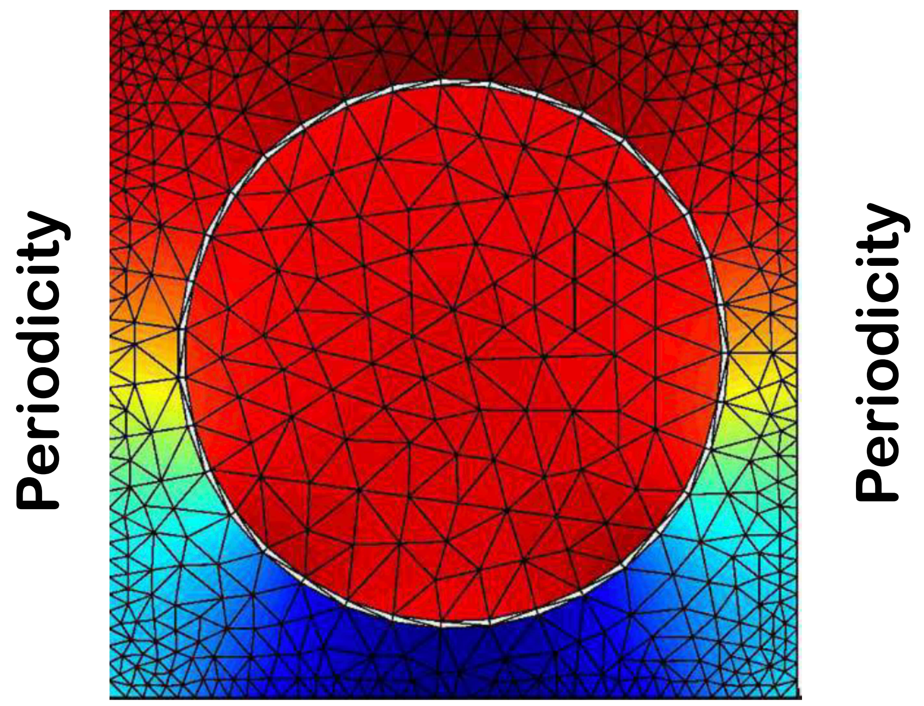
Case of the periodic structure

Extended state vector

$$\boldsymbol{\Sigma} = \begin{Bmatrix} \vdots \\ \boldsymbol{q}_{-1}^{+} \\ \boldsymbol{q}_{-1}^{-} \\ \boldsymbol{q}_{0}^{+} \\ \boldsymbol{q}_{0}^{-} \\ \boldsymbol{q}_{1}^{+} \\ \boldsymbol{q}_{1}^{-} \\ \vdots \end{Bmatrix}$$



Bloch decomposition on top and bottom interfaces

$$f(x) = \sum_{l \in \mathbb{Z}} f_l \, e^{-i[k_x + 2\pi l / D]x}$$

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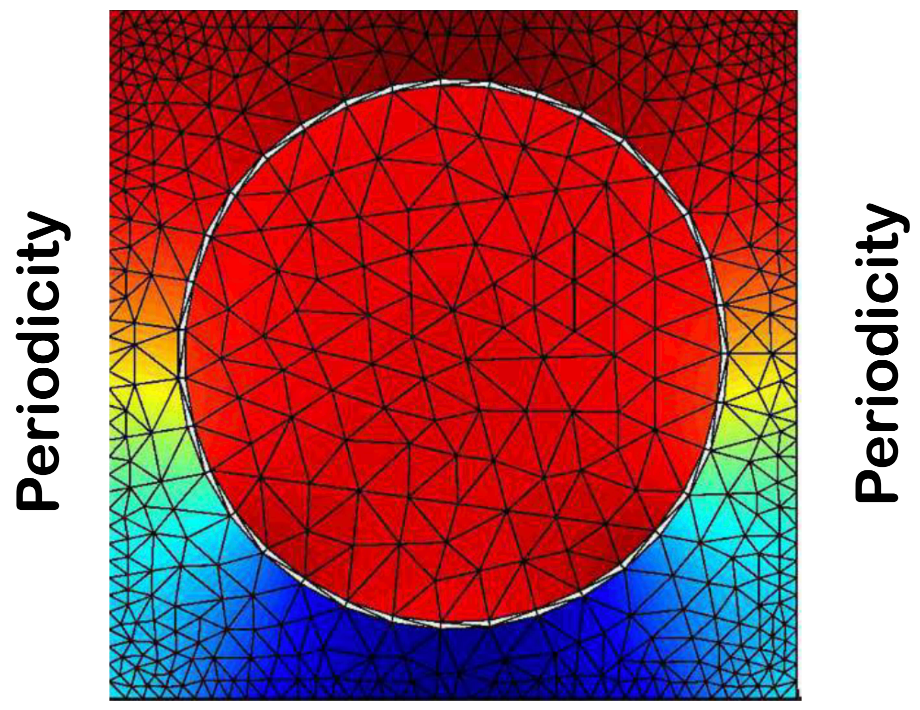

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After truncation

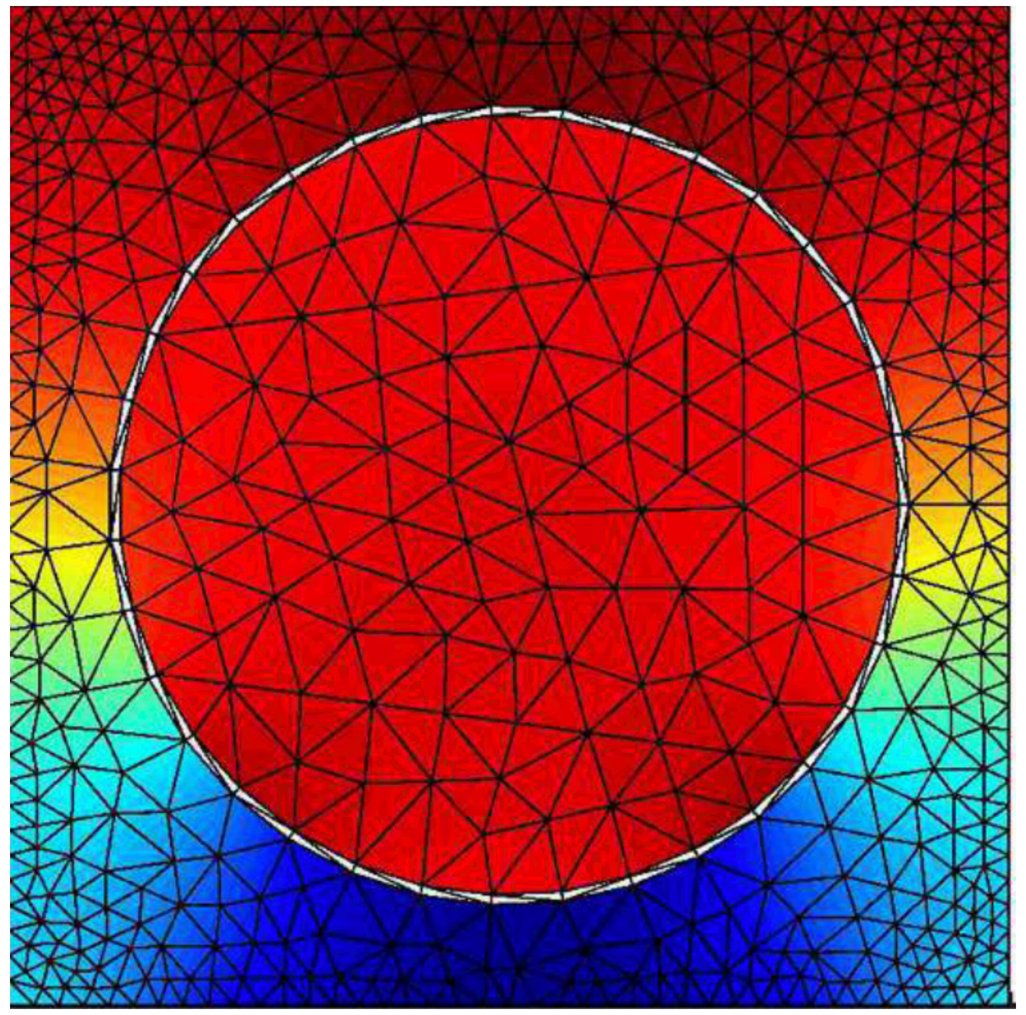
$$\begin{Bmatrix} \boldsymbol{\Sigma}^{b}_{-N} \\ \vdots \\ \boldsymbol{\Sigma}^{b}_{0} \\ \vdots \\ \boldsymbol{\Sigma}^{b}_{N} \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{T_{-N,-N}}] & \dots & [\boldsymbol{T_{-N,0}}] & \dots & [\boldsymbol{T_{-N,N}}] \\ [\boldsymbol{T_{0,-N}}] & \dots & [\boldsymbol{T_{0,0}}] & \dots & [\boldsymbol{T_{0,N}}] \\ [\boldsymbol{T_{N,-N}}] & \dots & [\boldsymbol{T_{N,0}}] & \dots & [\boldsymbol{T_{N,N}}] \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Sigma}^{t}_{-N} \\ \vdots \\ \boldsymbol{\Sigma}^{t}_{0} \\ \vdots \\ \boldsymbol{\Sigma}^{t}_{N} \end{Bmatrix}$$

- Homogeneous material : diagonal by block
- Non diagonal blacks : interaction of Bloch waves

Computation of the TM

Upper State vector

$2n_t$ $\boldsymbol{\Sigma_t}$



$\boldsymbol{\pi}$

FEM dofs

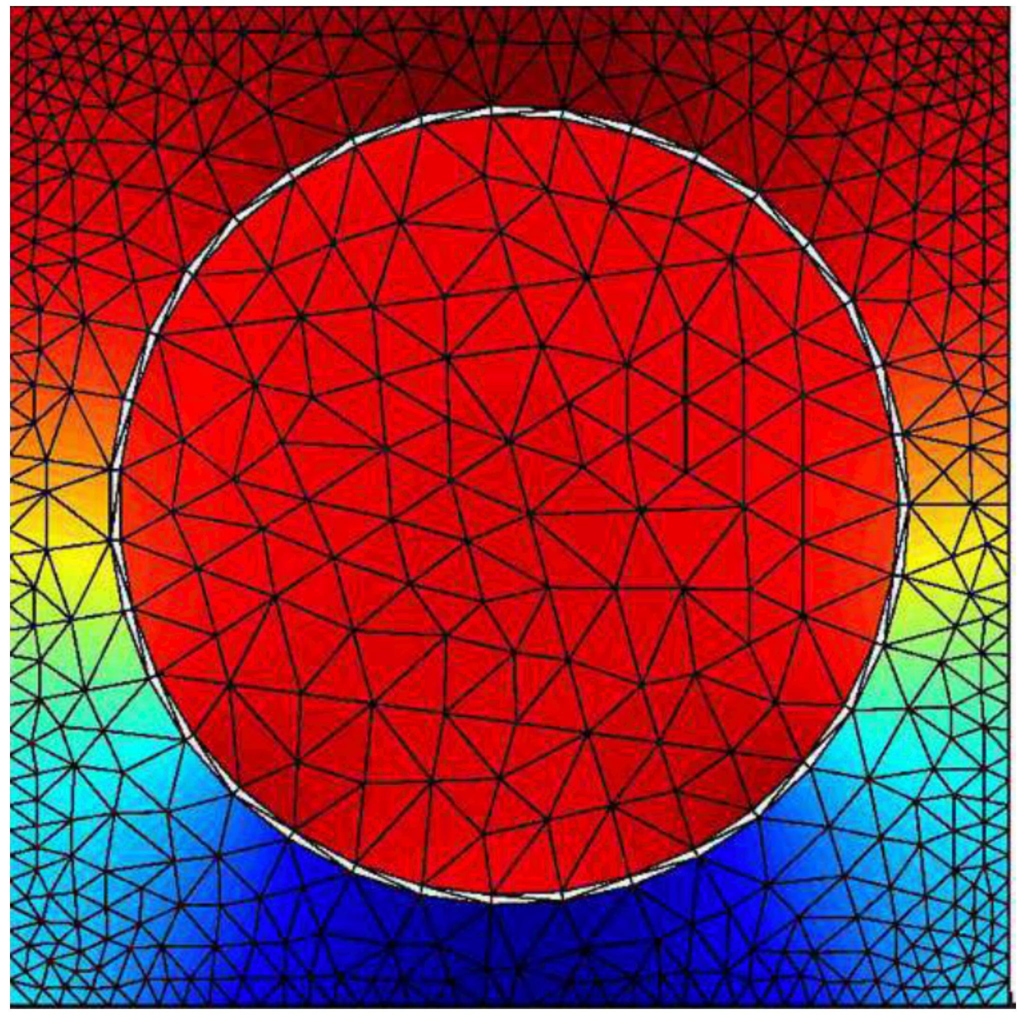
$2n_b$ $\boldsymbol{\Sigma_b}$

Lower State vector

Computation of the TM

Upper State vector

$2n_t$ $\boldsymbol{\Sigma_t}$



$\boldsymbol{\pi}$

FEM dofs

$2n_b$ $\boldsymbol{\Sigma_b}$

Lower State vector

Orthogonality relations

Stabilized weak forms

Orthogonality relations

$$\begin{bmatrix} [\boldsymbol{D_{tt}}] & [\boldsymbol{D_{ti}}] & [0] \\ [\boldsymbol{D_{it}}] & [\boldsymbol{D_{ii}}] & [\boldsymbol{D_{ib}}] \\ [0] & [\boldsymbol{D_{bi}}] & [\boldsymbol{D_{bb}}] \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Sigma}^t \\ \boldsymbol{\pi} \\ \boldsymbol{\Sigma}^b \end{Bmatrix} = \mathbf{0}.$$

Computation of the TM

Upper State vector

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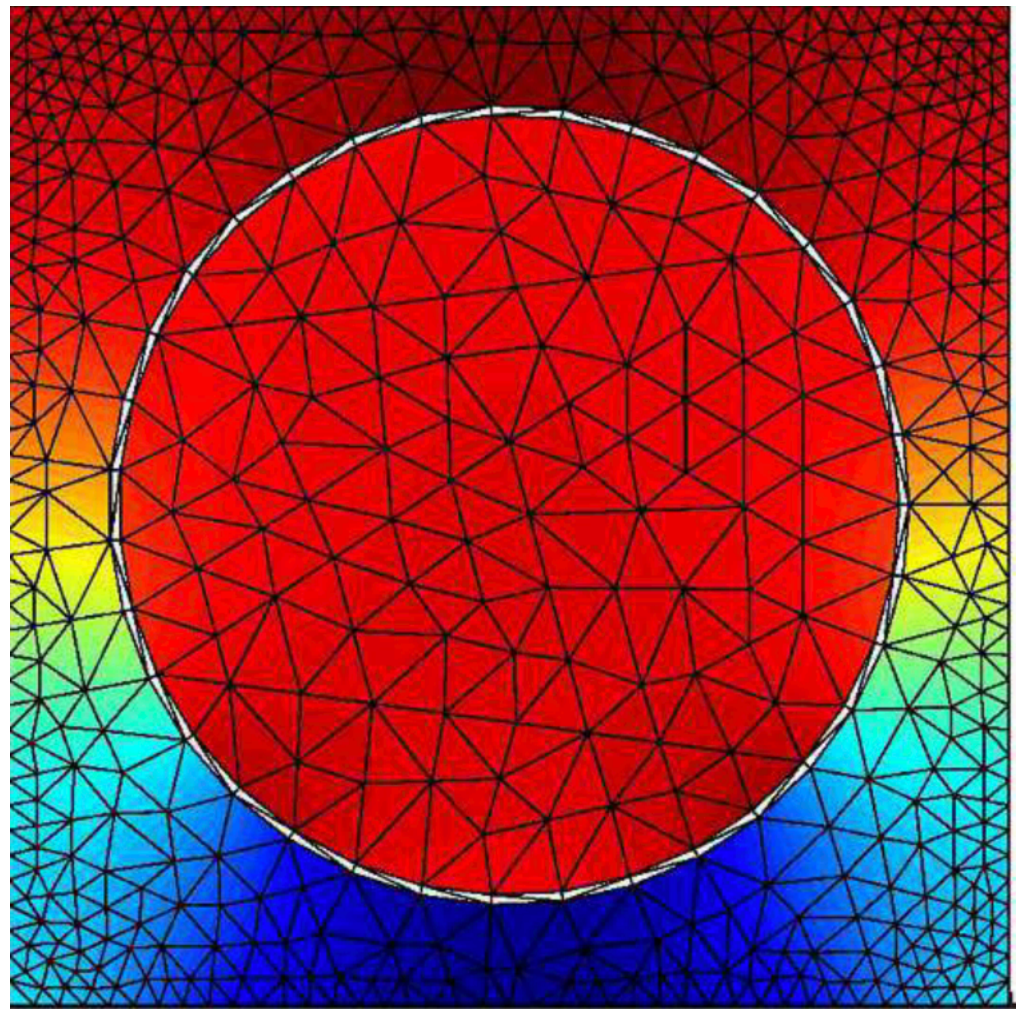

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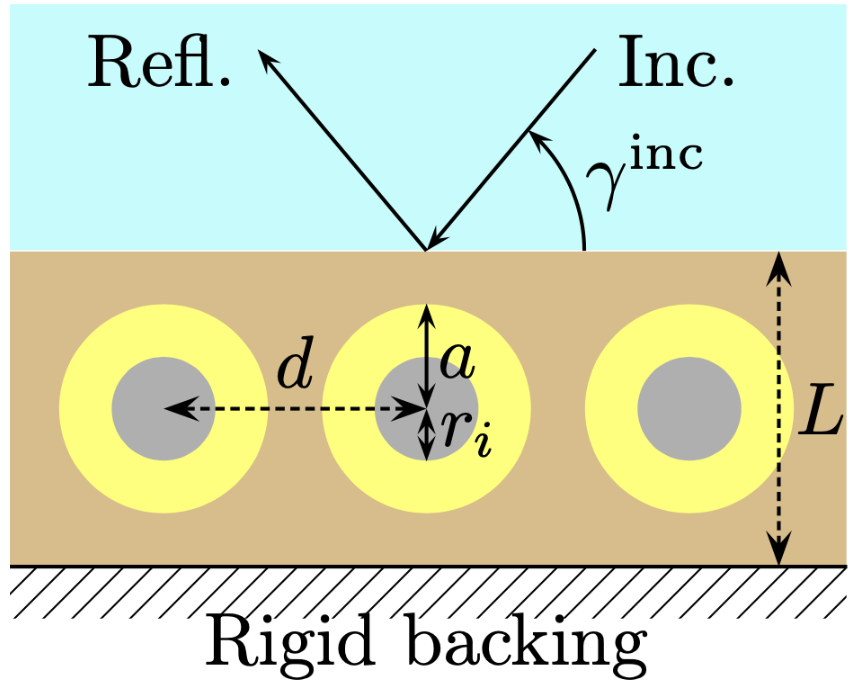
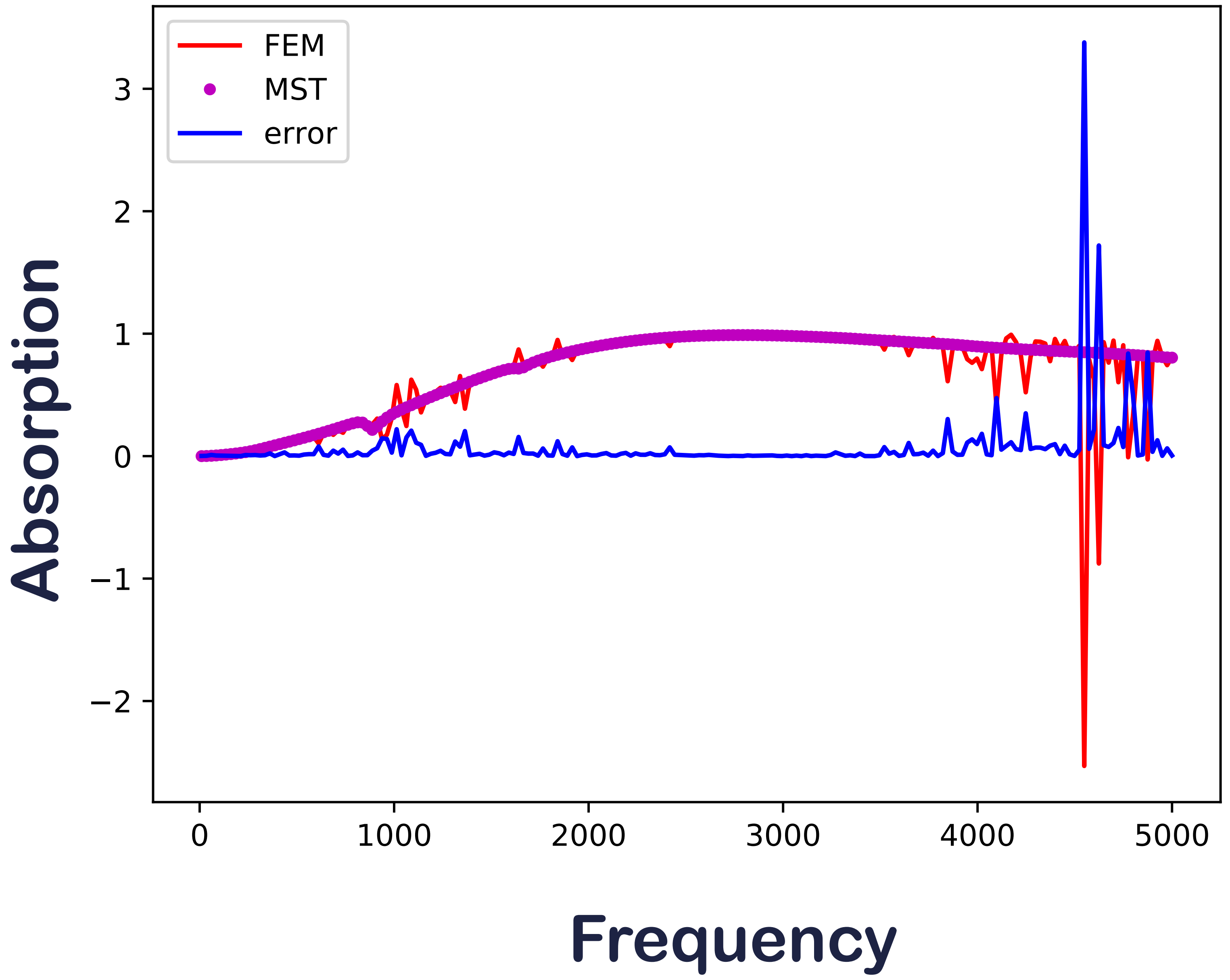
$$\boldsymbol{\Sigma}^b = -\begin{bmatrix} [D_{ti}][R_b] \\ [D_{bb}] + [D_{bi}][R_b] \end{bmatrix}^{-1} \begin{bmatrix} [D_{tt}] + [D_{ti}][R_t] \\ [D_{bi}][R_t] \end{bmatrix} \boldsymbol{\Sigma}^t$$

Application of the recursive approach
On the two matrices to stabilise

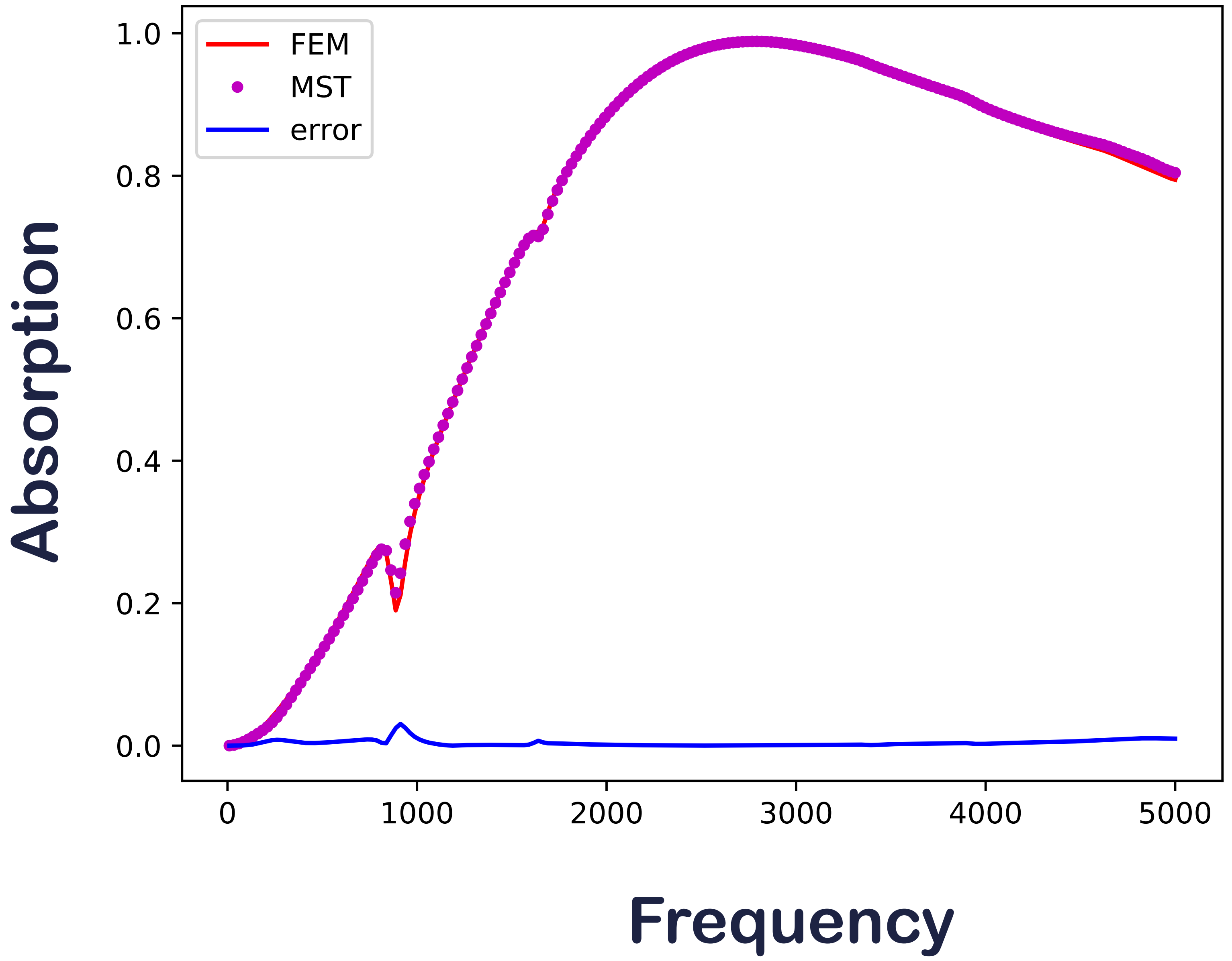
First: stability issues



Before



After



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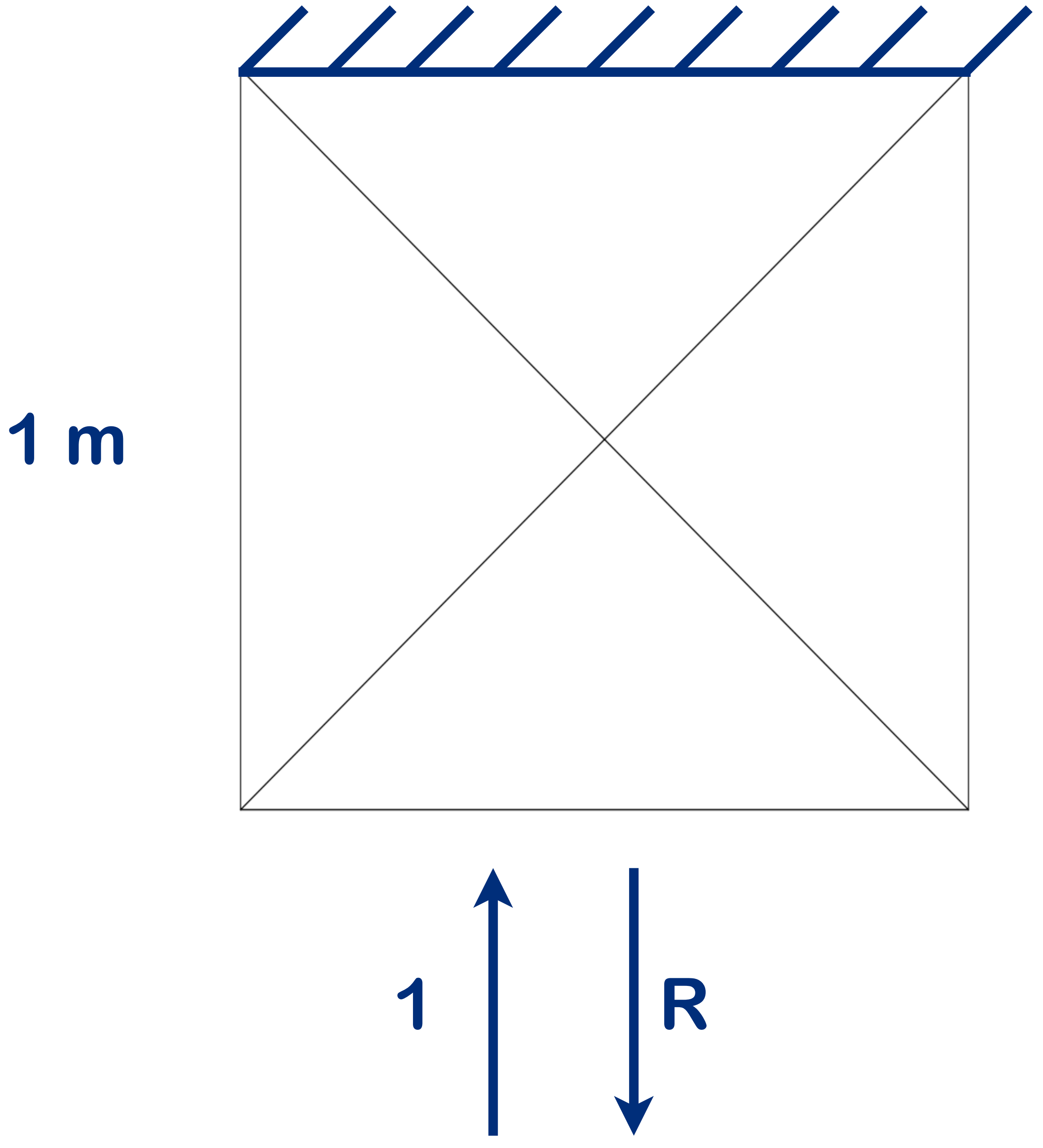
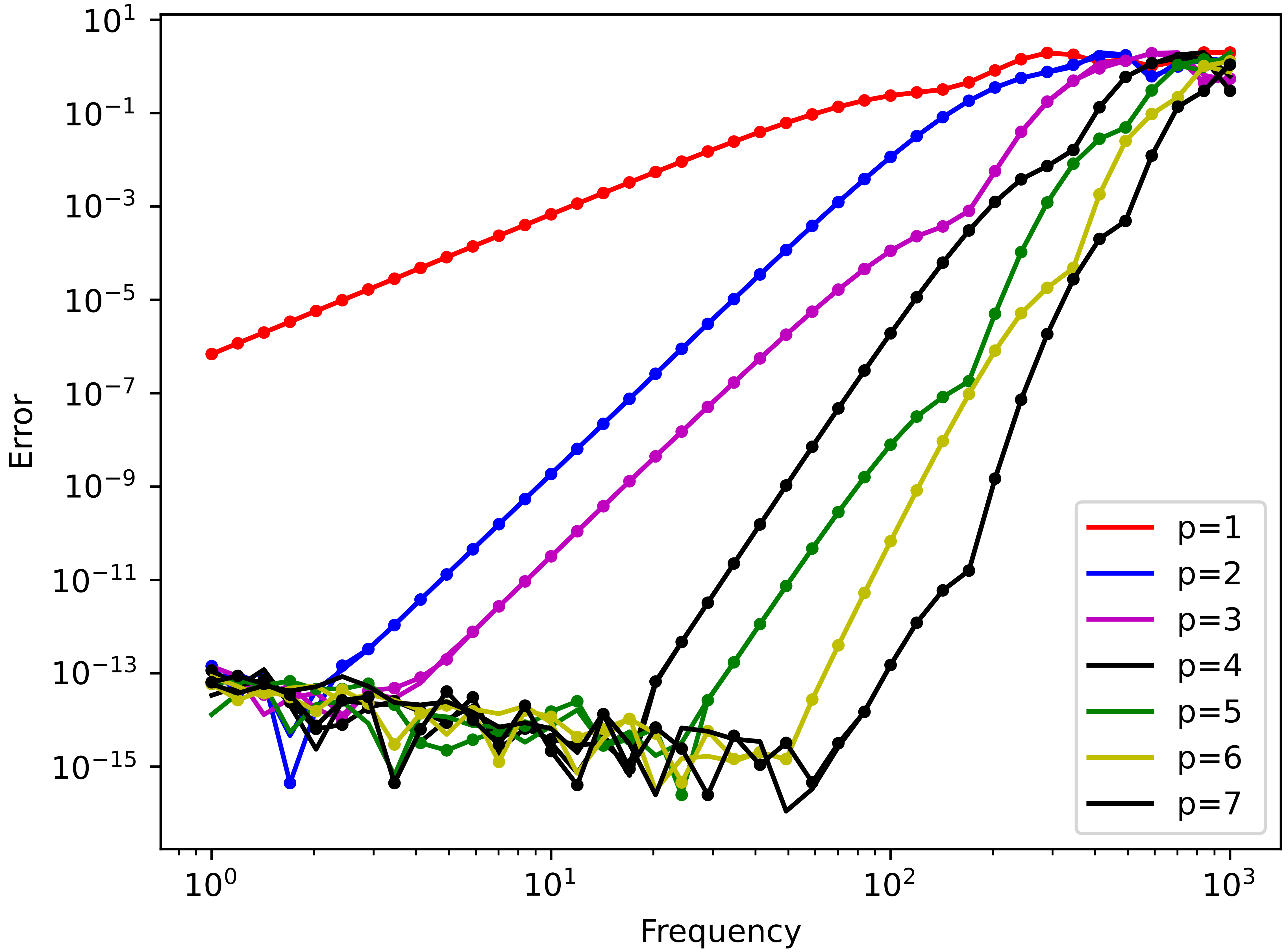
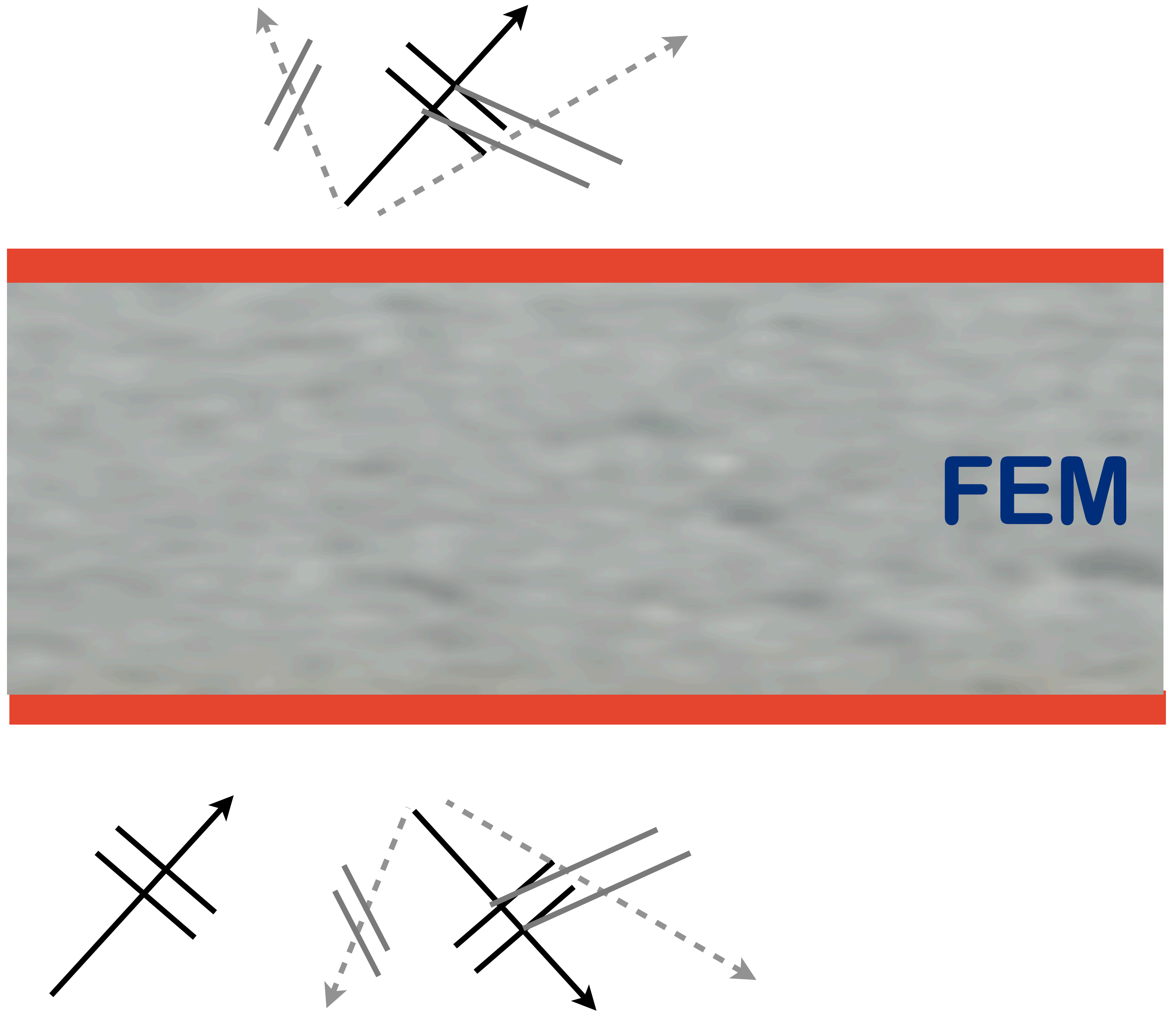
Performance of the method



- Air
- No Bloch waves

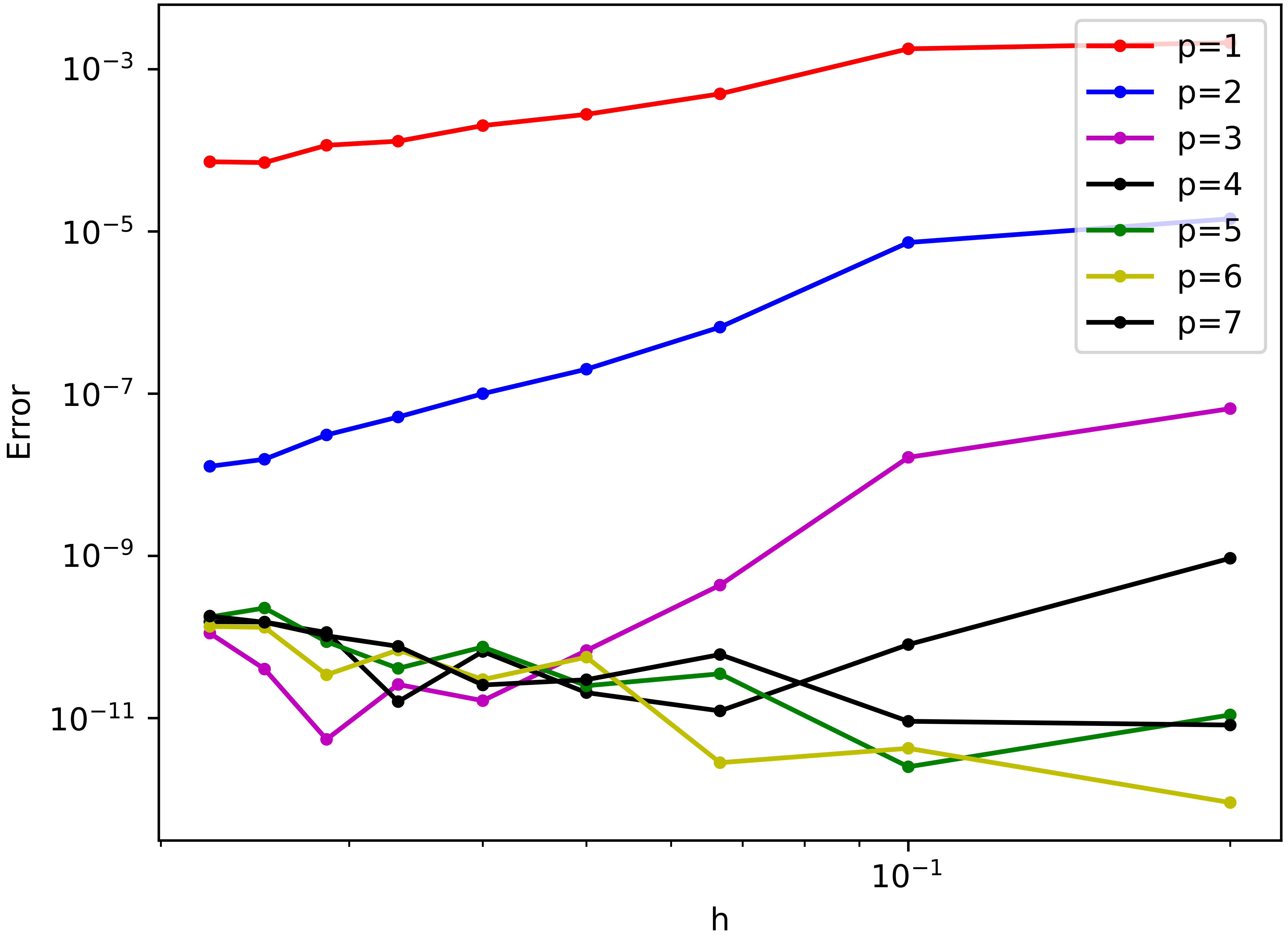


Multilayer case (sandwich panel)

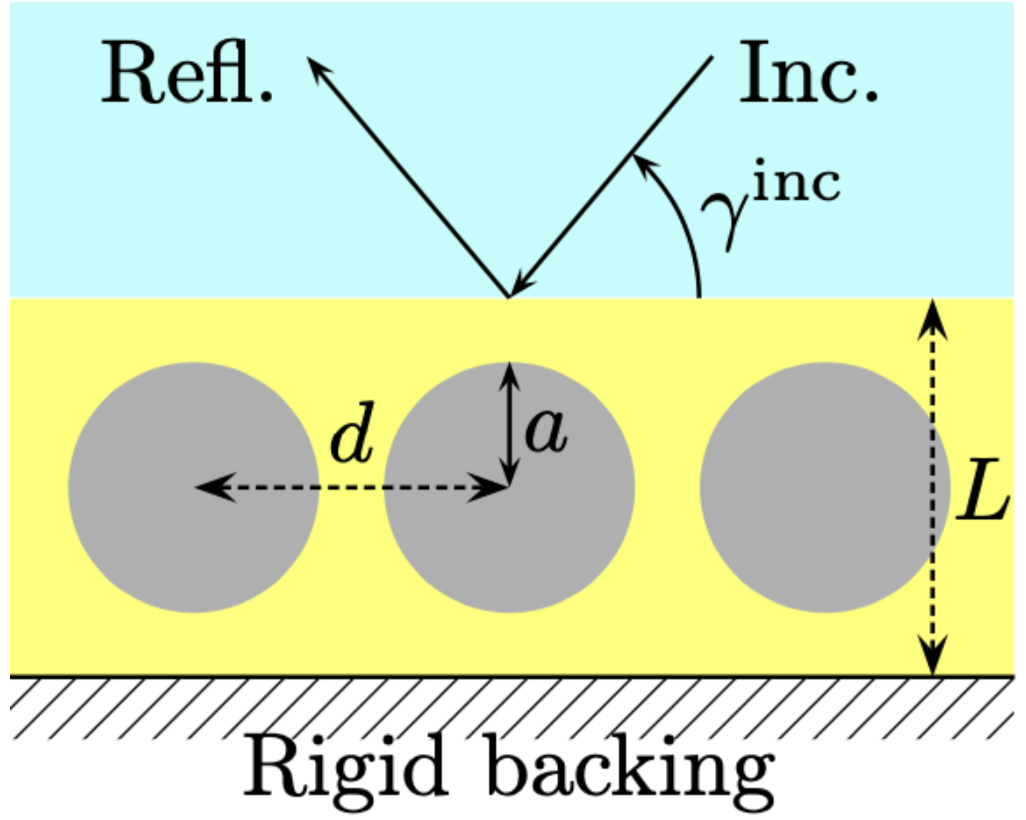


- Rubber 0.2 mm
- Melamine 2 cm
- Rubber 0.2 mm
- 45 ° incidence
- Period 20 cm

Influence of a mesh refinement

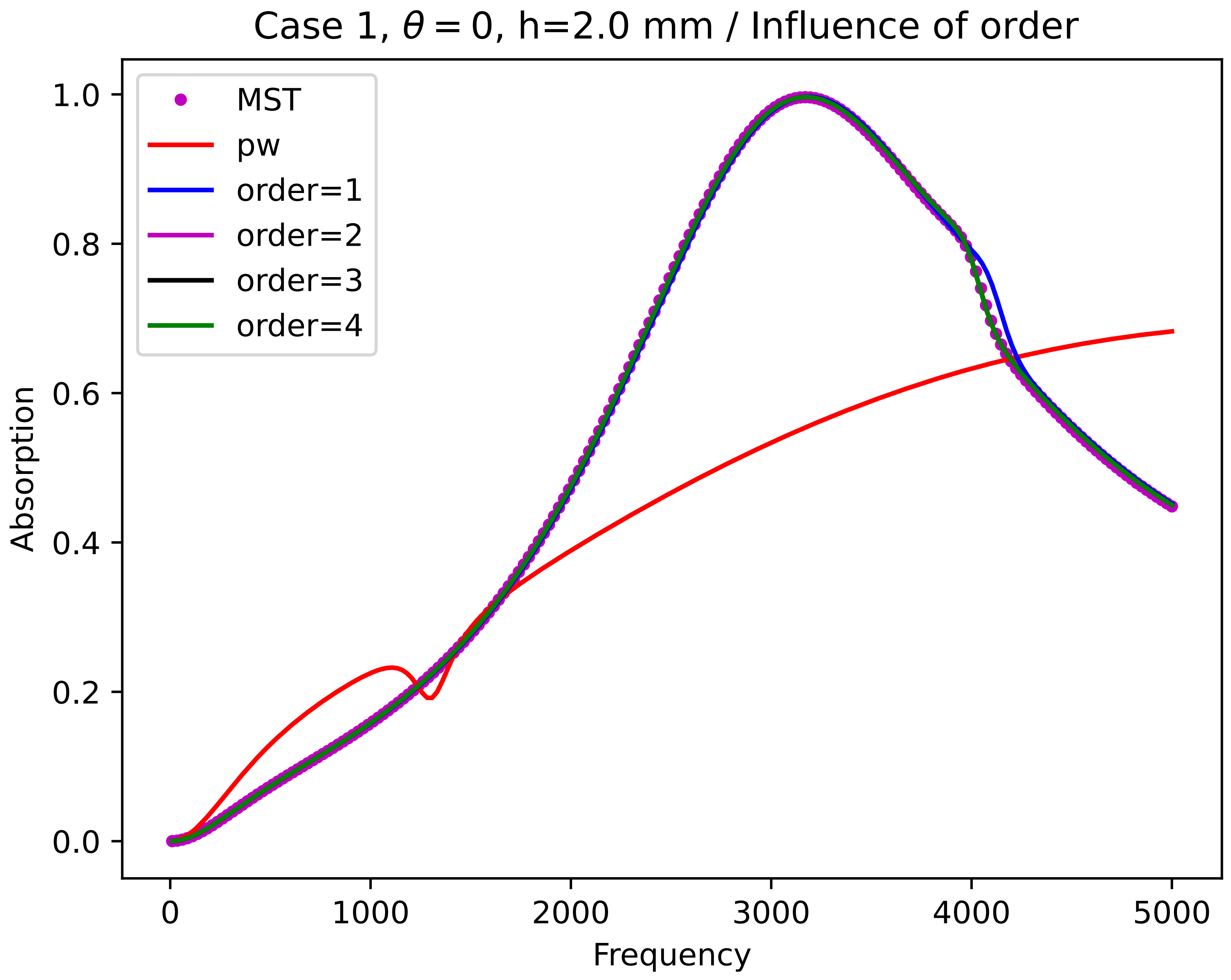


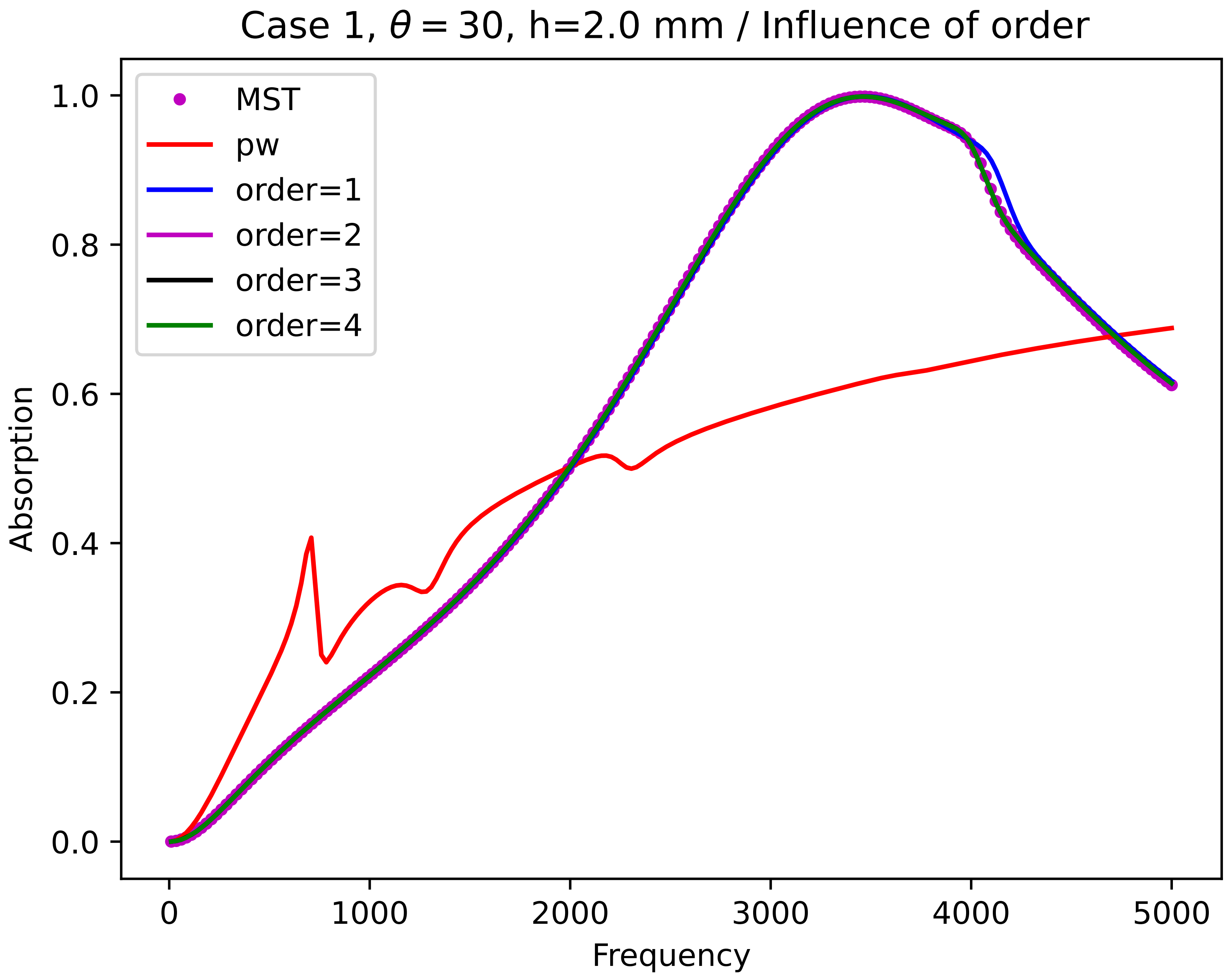
Comparaison MST



- **Poroelastic core (2cm x 2cm)**
- **Plastic inclusion**

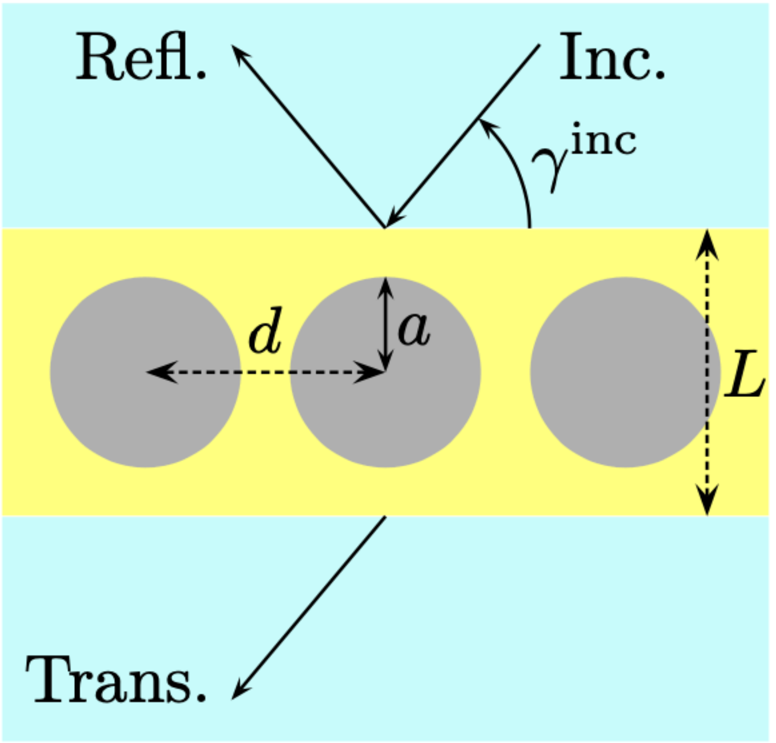
Weisser et al. JASA (2016)

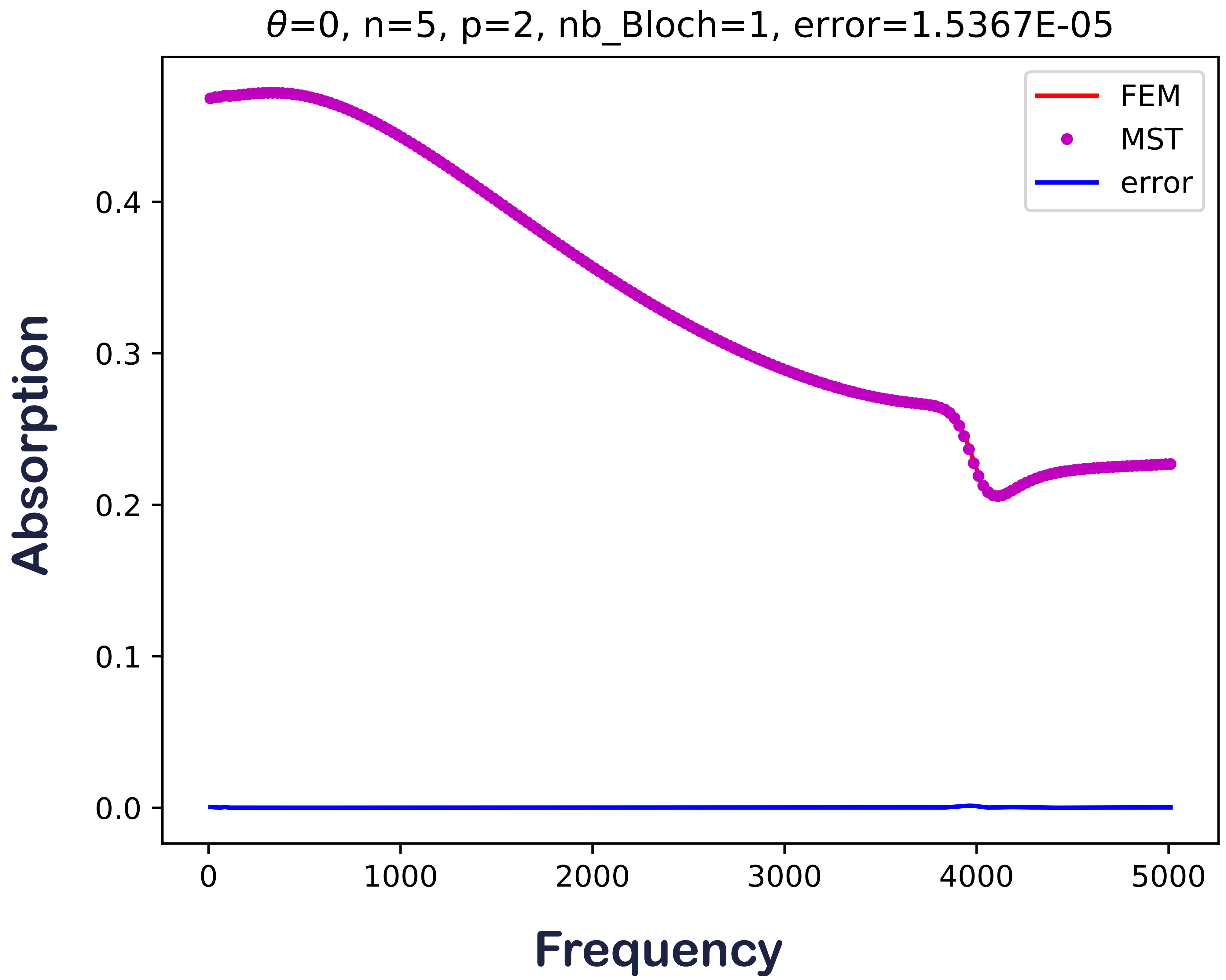




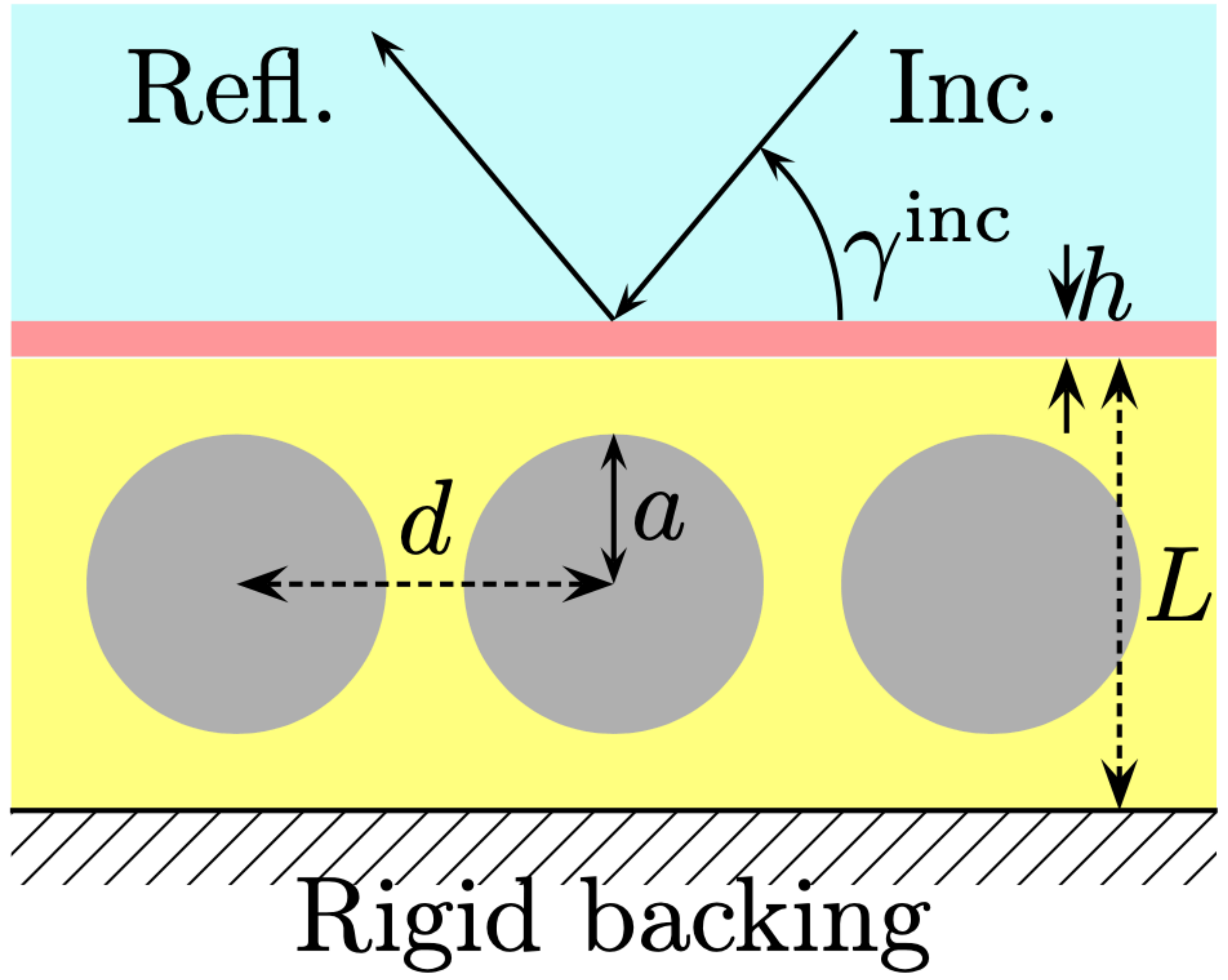
Case of transmission

- Case of transmission
- Poroelastic core (2cm x 2cm)
- Plastic inclusion
- Unpublished MST (at the moment)



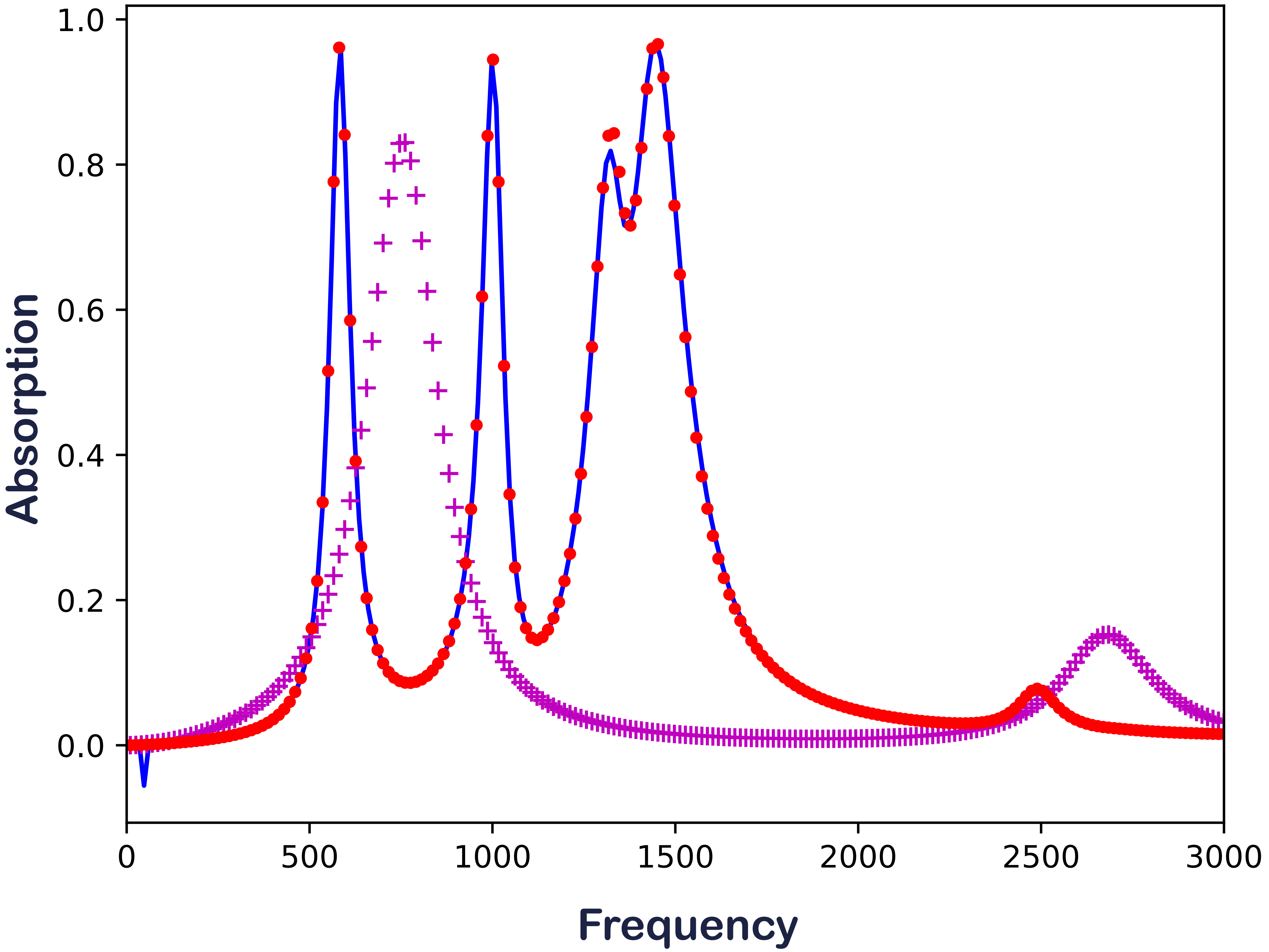


Another comparison with literature



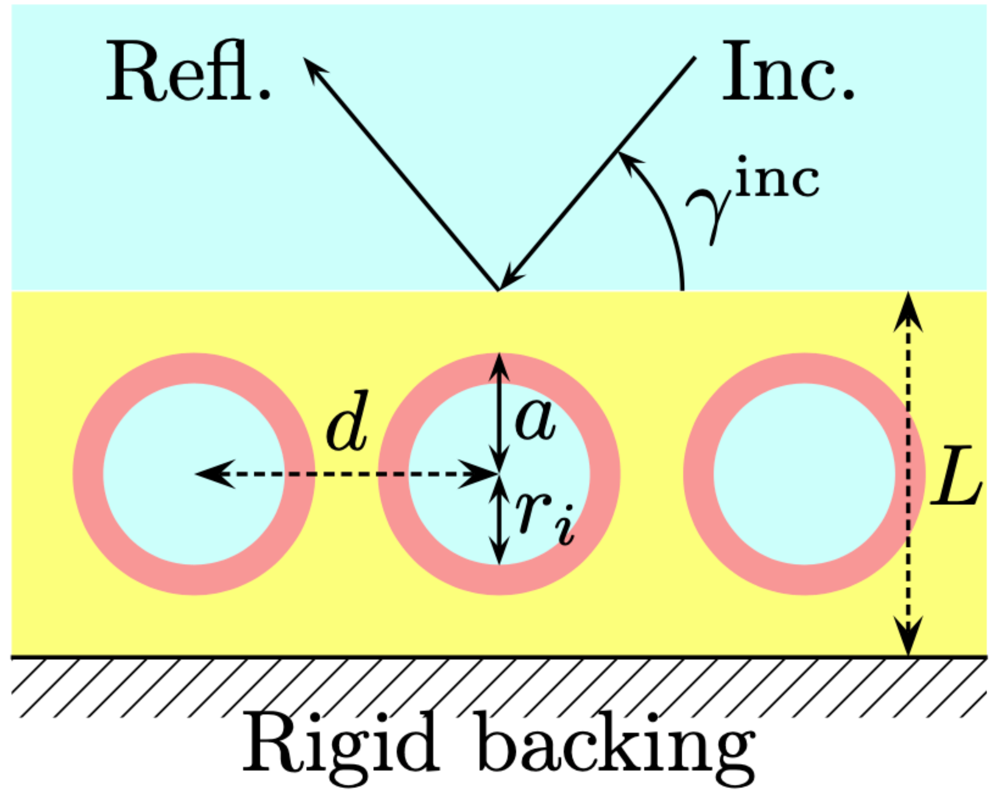
- Poroelastic core (2cm x 2cm)
- Elastic inclusion
- 0.2 mm rubber layer

Gaborit et al. AAA (2018)

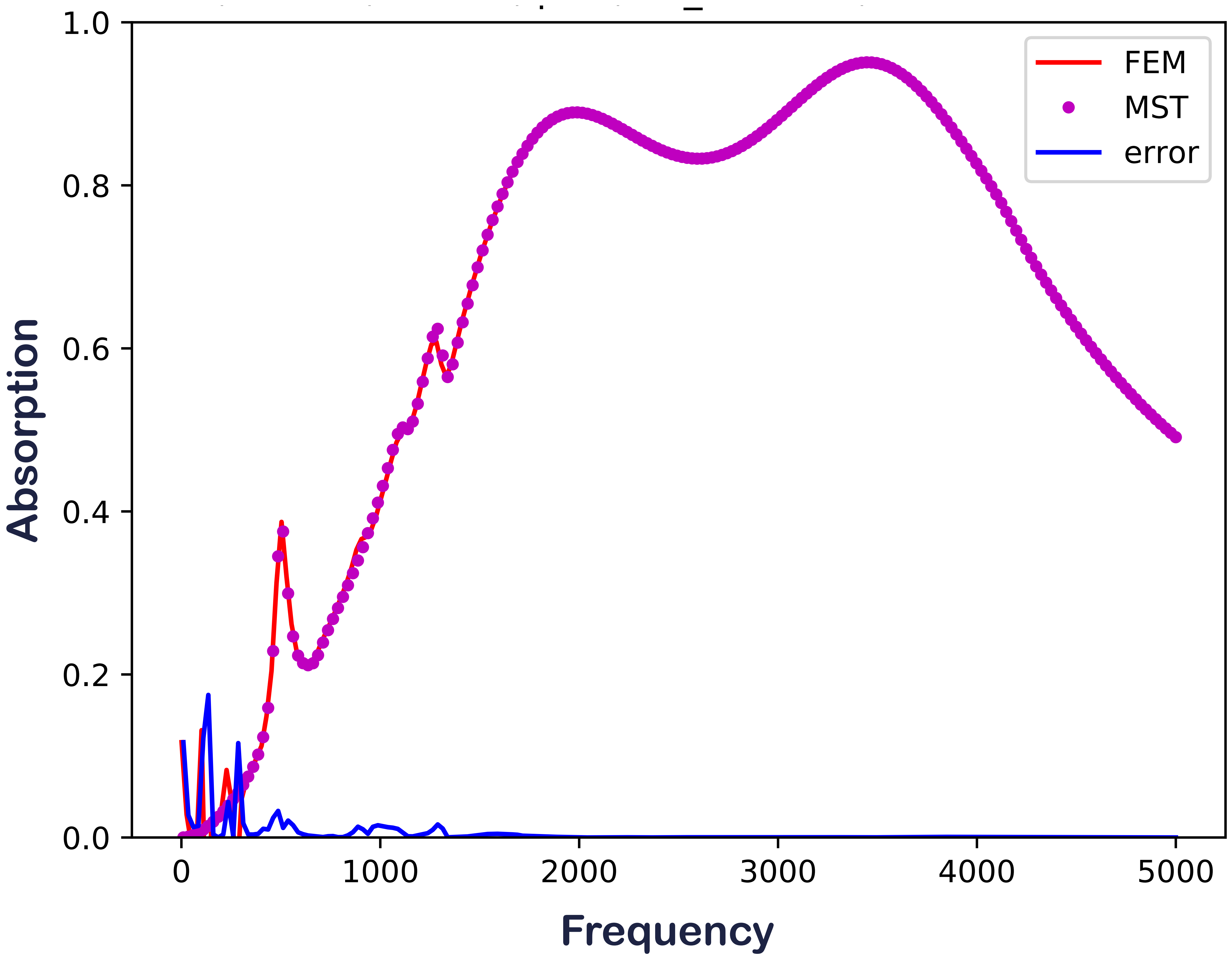


Limit of the method

Weisser et al. JASA (2016)

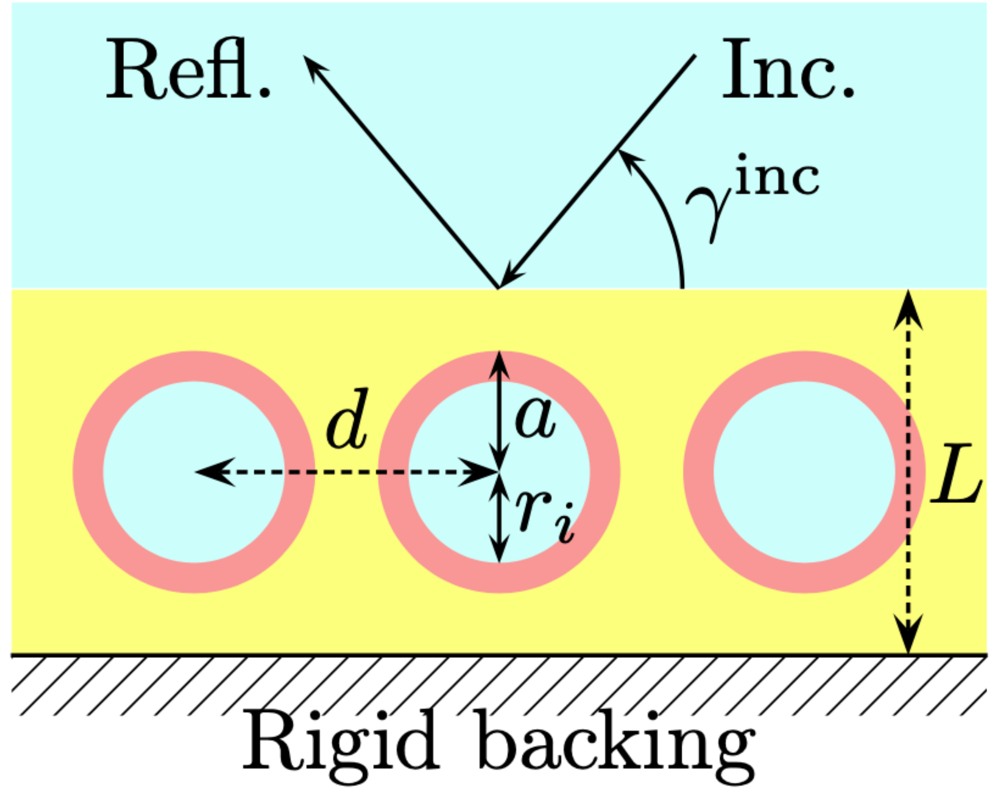


- Poroelastic core (2cm x 2cm)
- 0,2 mm rubber elastic skin
- 10 Bloch waves
- P = 7
- Conditioning issues at LF



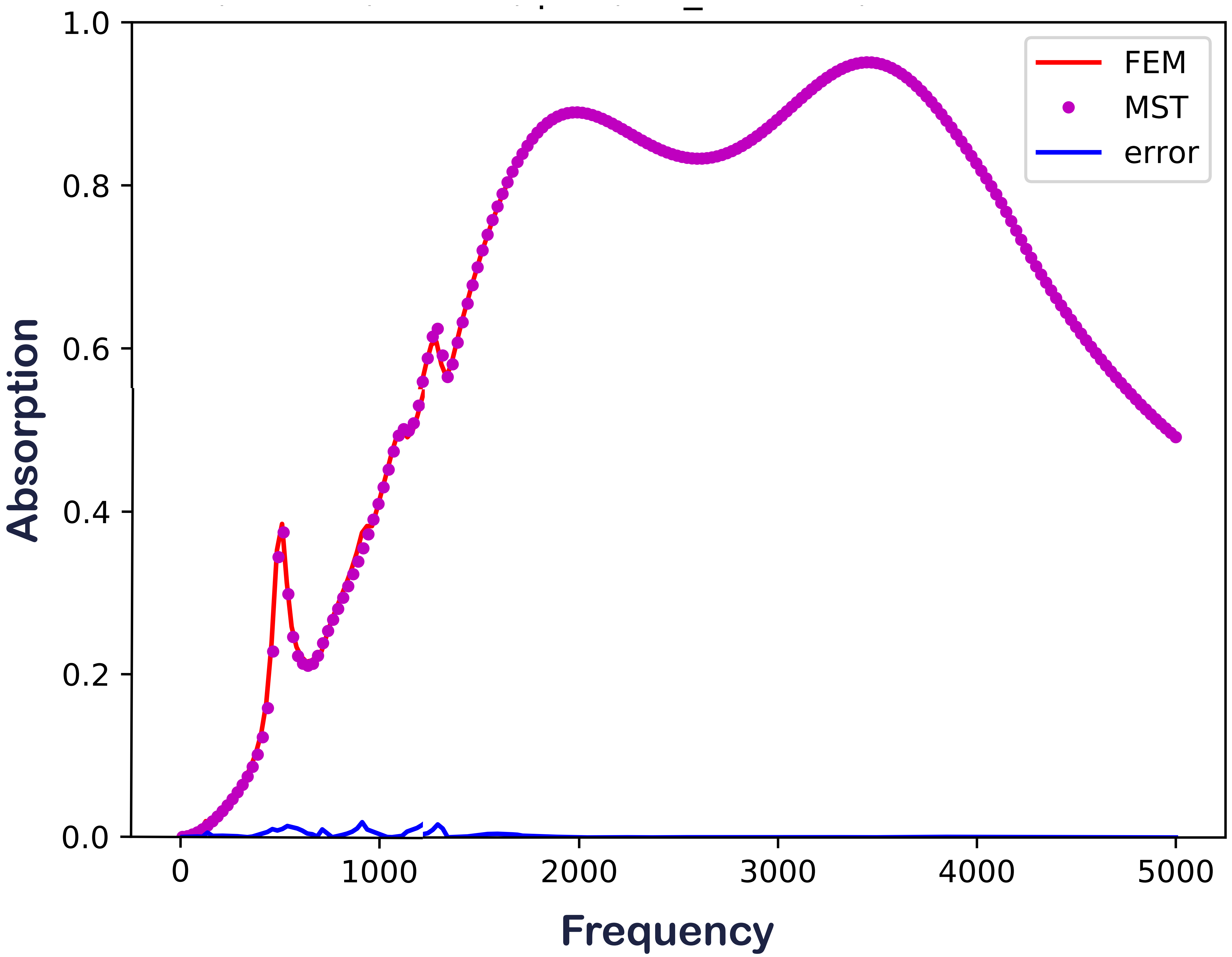
Limit of the method

Weisser et al. JASA (2016)



- **Poroelastic core (2cm x 2cm)**
- **0,2 mm rubber elastic skin**
- **10 Bloch waves**
- **P = 7**
- **Conditioning issues at LF**

Can be fixed by reducing the number of Bloch waves (p=2)



Conclusion

- Contributions
  - Reformulation of the recursive method in terms of characteristics
    - Information vector = characteristics
    - No a-priori in the interface relations
    - Valid at normal incidence
  - Stable approach for the periodic cell
    - Valid for higher order elements
    - Both on boundary operators and propagation in layers
    - Validated on various cases
- Limits / perspectives
  - Conditioning issues when the model is too discretised
  - Empirical fixes can be used
  - Theoretical alternatives can be found
  - Dispersion analysis

Institut d'Acoustique
Graduate School
Le Mans Université