Willis coupling in one-dimensional poroelastic laminates

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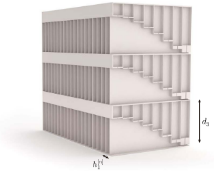
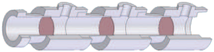
General motivation

- Various acoustic structures turns out to be asymmetric

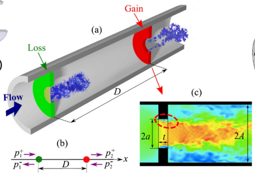


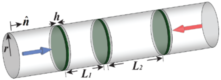
(Sci. Rep., 7: 13595, 2017)



(Phys. Rev. Lett., 104: 054301, 2010)

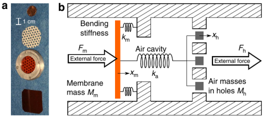
(Phys. Rev. Lett., 118: 174301, 2017)





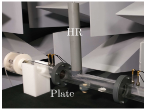
(Phys. Rev. Mater., 2: 125203, 2018)

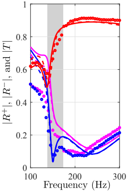
- Willis materials (Wave Motion, 3:1 , 1981)

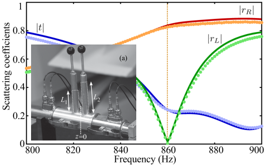


(Nat. Commun., 8: 15625, 2017)

(New J. Phys., 23: 053020, 2021)







(Phys. Rev. B, 98: 201102(R), 2018)

PHYSICAL REVIEW B **96**, 104303 (2017)

Origins of Willis coupling and acoustic bianisotropy in acoustic metamaterials through source-driven homogenization

Caleb F. Sieck,[1,2] Andrea Alù,[1] and Michael R. Haberman[3,2,*]

What are the specific features of Willis coupling in multiphase materials?

Governing equations

Assume a time dependence $e^{-\mathrm{i}\omega t}$ and a one-dimensional poroelastic medium

$$\left\{\begin{aligned} -\frac{\partial \sigma_{xx}}{\partial x} &= \omega^2 \rho_f w + \omega^2 \rho u, \\ -\frac{\bar{P}}{\partial x} &= \omega^2 \rho_f w + \omega^2 \tilde{\rho} w, \\ (K_G + \frac{4N}{3})\frac{\partial u}{\partial x} + \alpha M \frac{\partial w}{\partial x} &= \sigma_{xx}, \\ \alpha M \frac{\partial u}{\partial x} + M \frac{\partial w}{\partial x} &= \bar{P}, \end{aligned}\right.$$

where

- σ_{xx} is the normal stress
- $\bar{P} = -p$ is the pressure field with an opposite sign
- u is the elastic displacement
- $w = \phi(U - u)$, is the fluid/elastic relative displacement

(J. Appl. Phys.,33: 1482-1498, 1962; Geophysics, 56: 1950-1960, 1991)

Governing equations

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where

- ρ_f is the density of the saturating fluid
- $\rho = \phi \rho_f + (1 - \phi)\rho_s$ is the effective density of the poroelastic medium
- $\tilde{\rho}$ is a complex and frequency-dependent density
- K_G is the saturated (or undrained) modulus
- N is the dry shear modulus
- M is an additional elastic parameter
- α is an elastic coupling coefficient

(J. Fluid Mech., 176: 379-402, 1987; Veirteljahrsschr. Nat.forsch. Des. Zü.: 96: 1-23, 1951; J. Acoust. Soc. Am., 102: 1995-2006, 1997)

Solving the governing equations

Introducing the state vector $\boldsymbol{W} = < \bar{P}, \sigma_{xx}, w, u >^T$, the system is cast in the form

$$\frac{\partial}{\partial x}\boldsymbol{W} = \begin{bmatrix} 0 & 0 & -\omega^2\tilde{\rho} & -\omega^2\rho_f \\ 0 & 0 & -\omega^2\rho_f & -\omega^2\rho \\ \bar{C}_f & -\alpha\bar{C} & 0 & 0 \\ -\alpha\bar{C} & \bar{C} & 0 & 0 \end{bmatrix}\boldsymbol{W} = \boldsymbol{A}\boldsymbol{W},$$

with $\bar{C}_f = \dfrac{K_G + \frac{4N}{3}}{M(K_G + \frac{4N}{3}) - \alpha^2 M^2}$ and $\bar{C} = \dfrac{1}{(K_G + \frac{4N}{3}) - \alpha^2 M}$.

The solution is

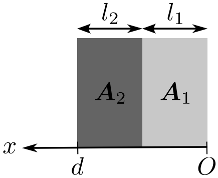
$$\boldsymbol{W}(l) = \boldsymbol{T}_l\boldsymbol{W}(0) = \mathrm{expm}\left(\boldsymbol{A}l\right)\boldsymbol{W}(0),$$

where $\mathrm{expm}\left(\boldsymbol{B}\right)$ is the matrix exponential of $\boldsymbol{B}$.

(Geophysics, 56: 1950-1960, 1991; J. Eng. Mech., 132: 519-531, 2006)

Derivation of the effective parameters

We assume a one-dimensional reciprocal and asymmetric system composed of a d-periodic repetition of a two-layer poroelastic unit-cell of respective propagation matrix $\boldsymbol{A}_e$.



$$\boldsymbol{W}(d) = \boldsymbol{T}_{l_2}\boldsymbol{T}_{l_1}\boldsymbol{W}(0) = \mathrm{expm}\left(\boldsymbol{A}_2 l_2\right)\mathrm{expm}\left(\boldsymbol{A}_1 l_1\right)\boldsymbol{W}(0) = \mathrm{expm}\left(\boldsymbol{A}_e d\right)\boldsymbol{W}(0).$$

The problem reduces to

$$\mathrm{expm}\left(\boldsymbol{A}_e d\right) = \mathrm{expm}\left(\boldsymbol{A}_2 l_2\right)\mathrm{expm}\left(\boldsymbol{A}_1 l_1\right),$$

the solution of which is the Baker-Campbell-Hausdorff formula

$$\boldsymbol{A}_e d = \underbrace{\boldsymbol{A}_2 l_2 + \boldsymbol{A}_1 l_1}_{\text{first-order homogenization}} + \underbrace{\frac{1}{2}\left(\boldsymbol{A}_2 l_2 \boldsymbol{A}_1 l_1 - \boldsymbol{A}_1 l_1 \boldsymbol{A}_2 l_2\right)}_{\text{Willis coupling}} + \cdots .$$

(J. Math. Phys., 3: 771-777, 1962; Proc. R. Soc. A: Math. Phys. Eng. Sci., 469: 20130240, 2013)

The effective parameters in details

The propagator matrix reads as

$$
\boldsymbol{A}_e^H \approx \left[\begin{array}{cc|cc} \multicolumn{2}{c}{\mathbf{0}} & -\omega^2 \tilde{\rho}_e & -\omega^2 \rho_{fe} \\ & & -\omega^2 \rho_{fe} & -\omega^2 \rho_e \\ \bar{C}_{fe} & -\alpha_e \bar{C}_e & \multicolumn{2}{c}{\mathbf{0}} \\ -\alpha_e \bar{C}_e & \bar{C}_e & & \end{array} \right],
$$

where

$$
\tilde{\rho}_e = \frac{\tilde{\rho}_1 l_1 + \tilde{\rho}_2 l_2}{d} \qquad \rho_{fe} = \frac{\rho_{f_1} l_1 + \rho_{f_2} l_2}{d} = \rho_f
$$

$$
\rho_e = \frac{\rho_1 l_1 + \rho_2 l_2}{d}
$$

$$
\bar{C}_{fe} = \frac{\bar{C}_{f_1} l_1 + \bar{C}_{f_2} l_2}{d} \qquad \alpha_e \bar{C}_e = \frac{\alpha_1 \bar{C}_1 l_1 + \alpha_2 \bar{C}_2 l_2}{d}
$$

$$
\bar{C}_e = \frac{\bar{C}_1 l_1 + \bar{C}_2 l_2}{d}
$$

to first-order.

The effective parameters in details

The propagator matrix reads as

$$\boldsymbol{A}_e \approx \begin{bmatrix} & -\mathrm{i}\omega \boldsymbol{\Psi}_e & \begin{matrix} -\omega^2 \tilde{\rho}_e & -\omega^2 \rho_{fe} \\ -\omega^2 \rho_{fe} & -\omega^2 \rho_e \end{matrix} \\ \begin{matrix} \bar{C}_{fe} & -\alpha_e \bar{C}_e \\ -\alpha_e \bar{C}_e & \bar{C}_e \end{matrix} & & \mathrm{i}\omega {\boldsymbol{\Psi}_e}^T \end{bmatrix},$$

where

$$\boldsymbol{\Psi}_e = \frac{-\mathrm{i}\omega l_1 l_2}{2d} \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{bmatrix},$$

with

$$\begin{aligned} \zeta_{11} &= \tilde{\rho}_1 \bar{C}_{f_2} - \tilde{\rho}_2 \bar{C}_{f_1} + \rho_{f_2} \alpha_1 \bar{C}_1 - \rho_{f_1} \alpha_2 \bar{C}_2 \\ \zeta_{12} &= \tilde{\rho}_2 \alpha_1 \bar{C}_1 - \tilde{\rho}_1 \alpha_2 \bar{C}_2 + \rho_{f_1} \bar{C}_2 - \rho_{f_2} \bar{C}_1 \\ \zeta_{21} &= \rho_2 \alpha_1 \bar{C}_1 - \rho_1 \alpha_2 \bar{C}_2 + \rho_{f_1} \bar{C}_{f_2} - \rho_{f_2} \bar{C}_{f_1} \\ \zeta_{22} &= \rho_{f_2} \alpha_1 \bar{C}_1 - \rho_{f_1} \alpha_2 \bar{C}_2 + \rho_1 \bar{C}_2 - \rho_2 \bar{C}_1 \end{aligned}$$

including second-order terms.

(J. Mech. Phys. Solids, 77: 158-178, 2015)

Effect of the viscous losses (Darcy's law)

We assume the skeleton is motionless. The state vector becomes $\mathbf{W}^{ef} =< p, \mathcal{V} >=< p, \phi\dot{U} >$ and the equation of motion reduces to

$$\frac{\partial}{\partial x}\mathbf{W}^{ef} = \mathrm{i}\omega \begin{bmatrix} -\Psi_e^{ef} & \tilde{\rho}_e^{ef} \\ \bar{C}_{fe}^{ef} & \Psi_e^{ef} \end{bmatrix} \mathbf{W}^{ef} = \mathbf{A}_e^{ef}\mathbf{W}^{ef}$$

with

$$\Psi_e^{ef} = \frac{\mathrm{i}\omega l_1 l_2}{d}\left(\tilde{\rho}_1\tilde{C}_{f2} - \tilde{\rho}_2\tilde{C}_{f1}\right) \underset{\omega\to 0}{\longrightarrow} \frac{-l_1 l_2 \eta}{d P_0}\left(\frac{\phi_2}{\kappa_{01}} - \frac{\phi_1}{\kappa_{02}}\right)$$

$$\tilde{\rho}_e^{ef} = \frac{\tilde{\rho}_1 l_1 + \tilde{\rho}_2 l_2}{d} \underset{\omega\to 0}{\longrightarrow} \frac{\mathrm{i}\eta}{d\omega}\left(\frac{l_1}{\kappa_{01}} + \frac{l_2}{\kappa_{02}}\right),$$

$$\bar{C}_{fe}^{ef} = \frac{\bar{C}_{f1} l_1 + \bar{C}_{f2} l_2}{d} \underset{\omega\to 0}{\longrightarrow} \frac{1}{d P_0}\left(\phi_1 l_1 + \phi_2 l_2\right),$$

where κ_{0j}, $j = 1, 2$, are the viscous permeabilities and P_0 and η are parameters of the saturating fluid.

The Willis coupling does not vanish at low frequency!

(Wave Motion, 110: 102892, 2022)

Effect of the viscous losses (Darcy's law)

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$$\frac{\partial}{\partial x}\boldsymbol{W}^{ef} = \mathrm{i}\omega \begin{bmatrix} -\Psi_e^{ef} & \tilde{\rho}_e^{ef} \\ \bar{C}_{fe}^{ef} & \Psi_e^{ef} \end{bmatrix} \boldsymbol{W}^{ef} = \boldsymbol{A}_e^{ef}\boldsymbol{W}^{ef}$$

BUT

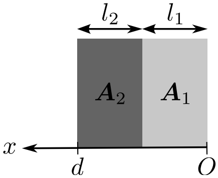
$$k_e^{ef} = \sqrt{\tilde{\rho}_e^{ef}\,\bar{C}_{fe}^{ef} + \left(\Psi_e^{ef}\right)^2} \quad \underset{\omega\to 0}{\longrightarrow} \quad k_e^H = \sqrt{\tilde{\rho}_e^{ef}\,\bar{C}_{fe}^{ef}}$$

$$Z_e^{\pm} = \tilde{\rho}_e^{ef} / \left(\sqrt{\tilde{\rho}_e^{ef}\,\bar{C}_{fe}^{ef} + \left(\Psi_e^{ef}\right)^2} \mp \Psi_e^{ef}\right) \quad \underset{\omega\to 0}{\longrightarrow} \quad Z_e^H = \sqrt{\tilde{\rho}_e^{ef} / \bar{C}_{fe}^{ef}},$$

The laminate structure falls back to symmetric at low frequency when viscous losses are accounted for, although the Willis coupling does not vanish!

Numerical example and validation

We consider a laminated poroelastic structure composed of two air-saturated poroelastic layers. The layer thicknesses are $l_1 = l_2 = 1$ cm, such that $d = l_1 + l_2 = 2$ cm.



	ϕ	τ_∞	λ (μm)	λ' (μm)	k_0 (m^2)	k_0' (m^2)	$K_b(1+i\zeta)$ (kPa)	ν	ρ_s
M1	0.95	1.1	15	45	4.4×10^{-10}	5.3×10^{-10}	$445 - i22$	0.24	2520
M2	0.96	2.2	110	352	2×10^{-9}	2.5×10^{-9}	$83 - i4$	0.21	925

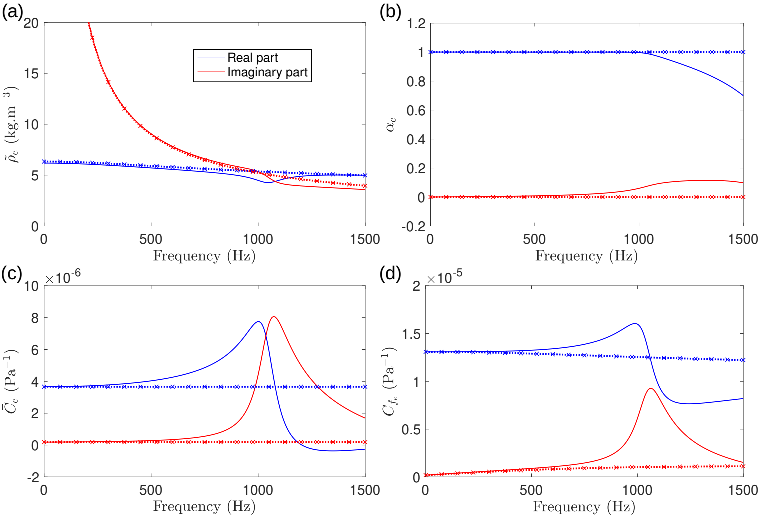
Once the total transfer matrix $\boldsymbol{T}_d = \text{expm}\left(\boldsymbol{A}_e^{num} d\right)$ is calculated, we immediately end up with

$$\boldsymbol{A}_e^{num} = \frac{1}{d}\text{logm}\left(\boldsymbol{T}_d\right) = \frac{1}{d}\boldsymbol{V}\,\mathbf{diag}(\log(\Lambda^\pm))\,\boldsymbol{V}^{-1},$$

where $\Lambda^\pm$ are the eigenvalues of $\boldsymbol{T}_d$ and $\boldsymbol{V}$ the eigenvector matrix.

(Proc. R. Soc. A, 467: 1749-69, 2011)

The effective properties



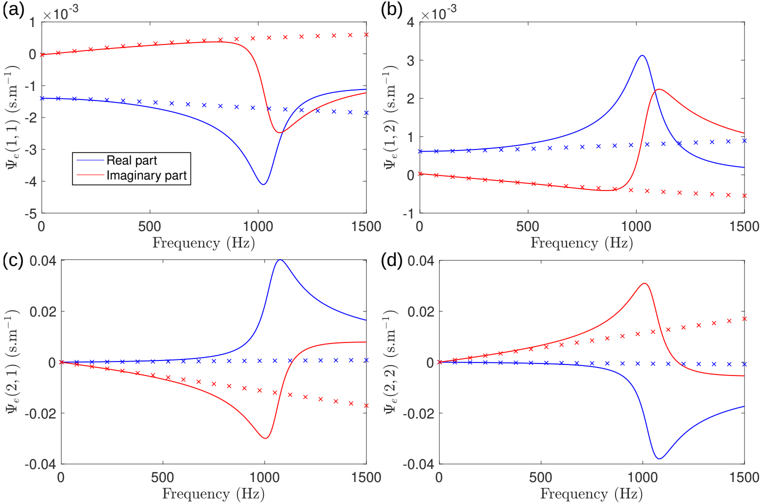
solid line: direct numerical evaluation

dashed line: first-order homogenization

marker: second-order homogenization

Closed-form expressions are valid up to α_e deviates from 1.

The effective properties

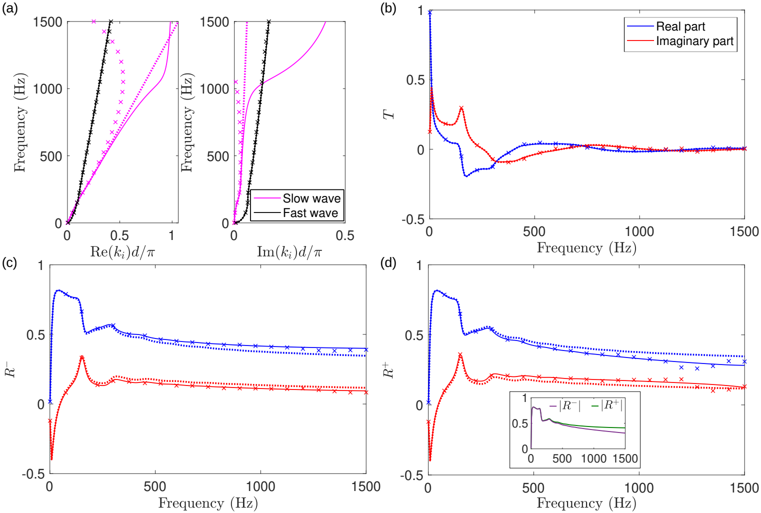


solid line: direct numerical evaluation

marker: second-order homogenization

Willis coupling does not vanish at low frequency.

The scattering coefficients



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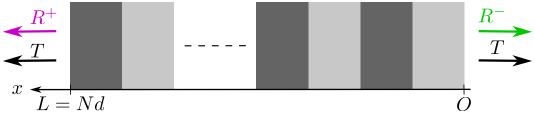
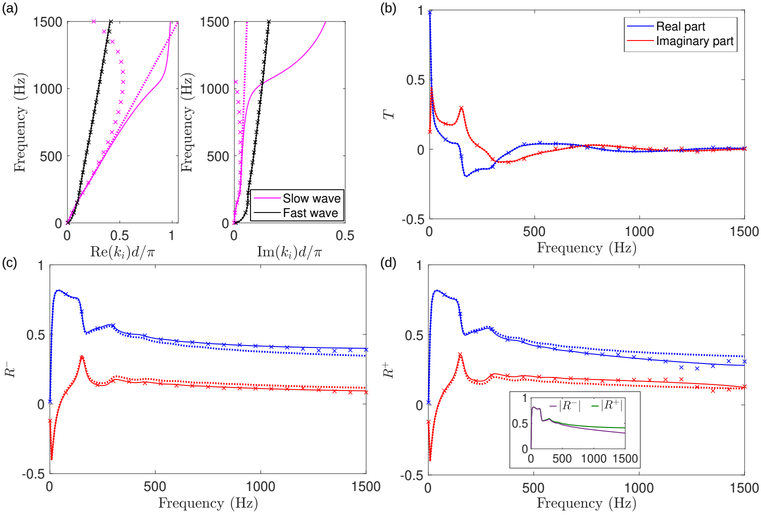


Scattering coefficients of a $L = 10d$ structure

The scattering coefficients



solid line: direct numerical evaluation

dashed line: first-order homogenization

marker: second-order homogenization

Willis coupling has to be accounted for.
The structure falls back to symmetric at low frequency.

Conclusion and perspectives

- Closed form expressions of the effective properties of a two-layer poroelastic unit-cell material, including Willis coupling, are derived via the Baker-Campbell-Hausdorff formula
    - Closed form expressions are valid up to α_e deviates from 1 when satuated by light fluid
    - Willis coupling do not vanish at low frequency because of the Darcy's law
    - The asymmetric structure falls back to symmetric at low frequency thanks to the Darcy's law
- Frequency range of validity of the scattering coefficients is wider when the Willis coupling matrix is taken into account than in its absence
- Acoustic wave control by multiphase asymmetric materials

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Please visit the website: https://sil-et-add.fr/en

Thank you for your attention!

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