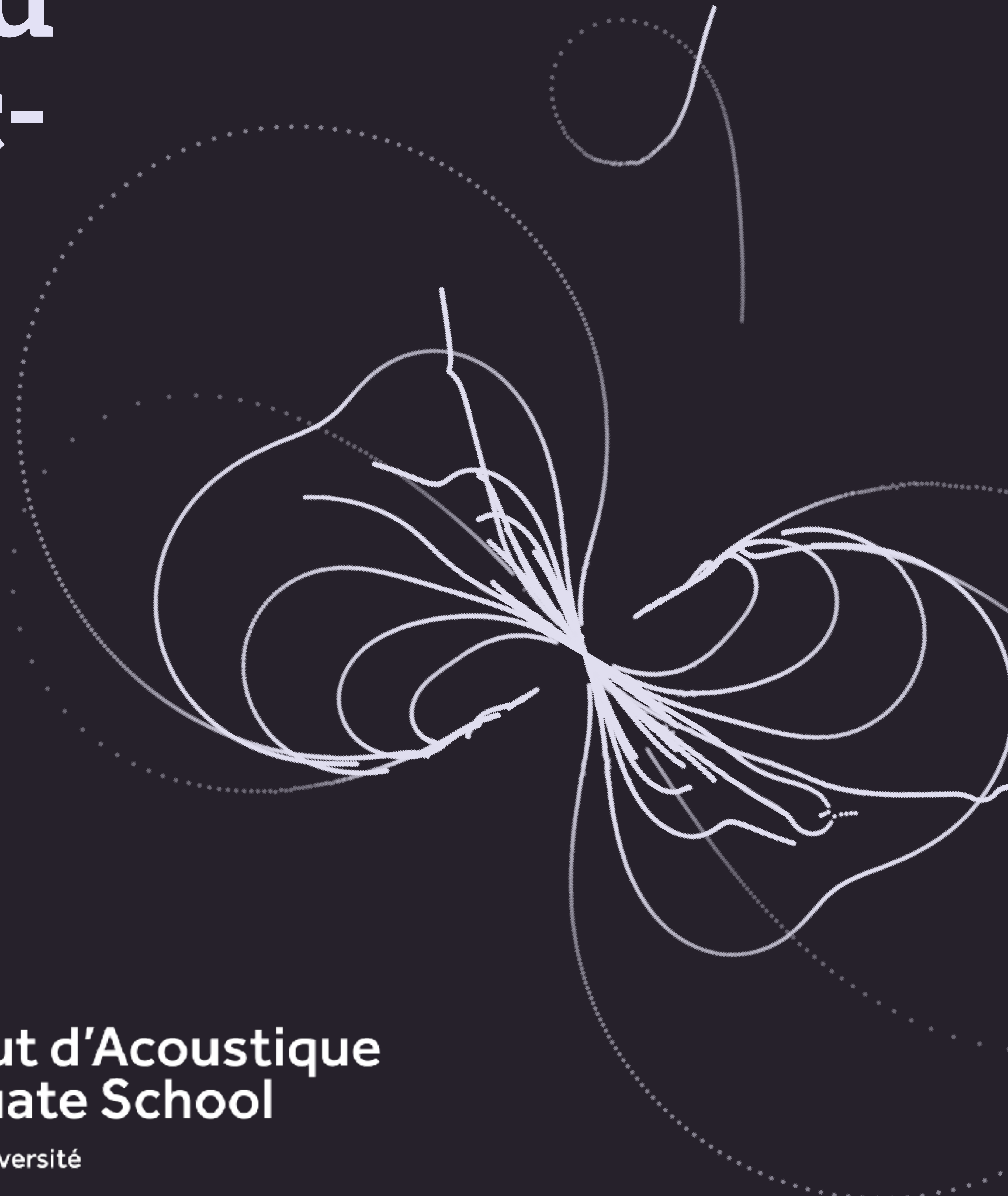
Dispersion relations of guided acoustic waves in poroelastic-based multilayer structures



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In collaboration with

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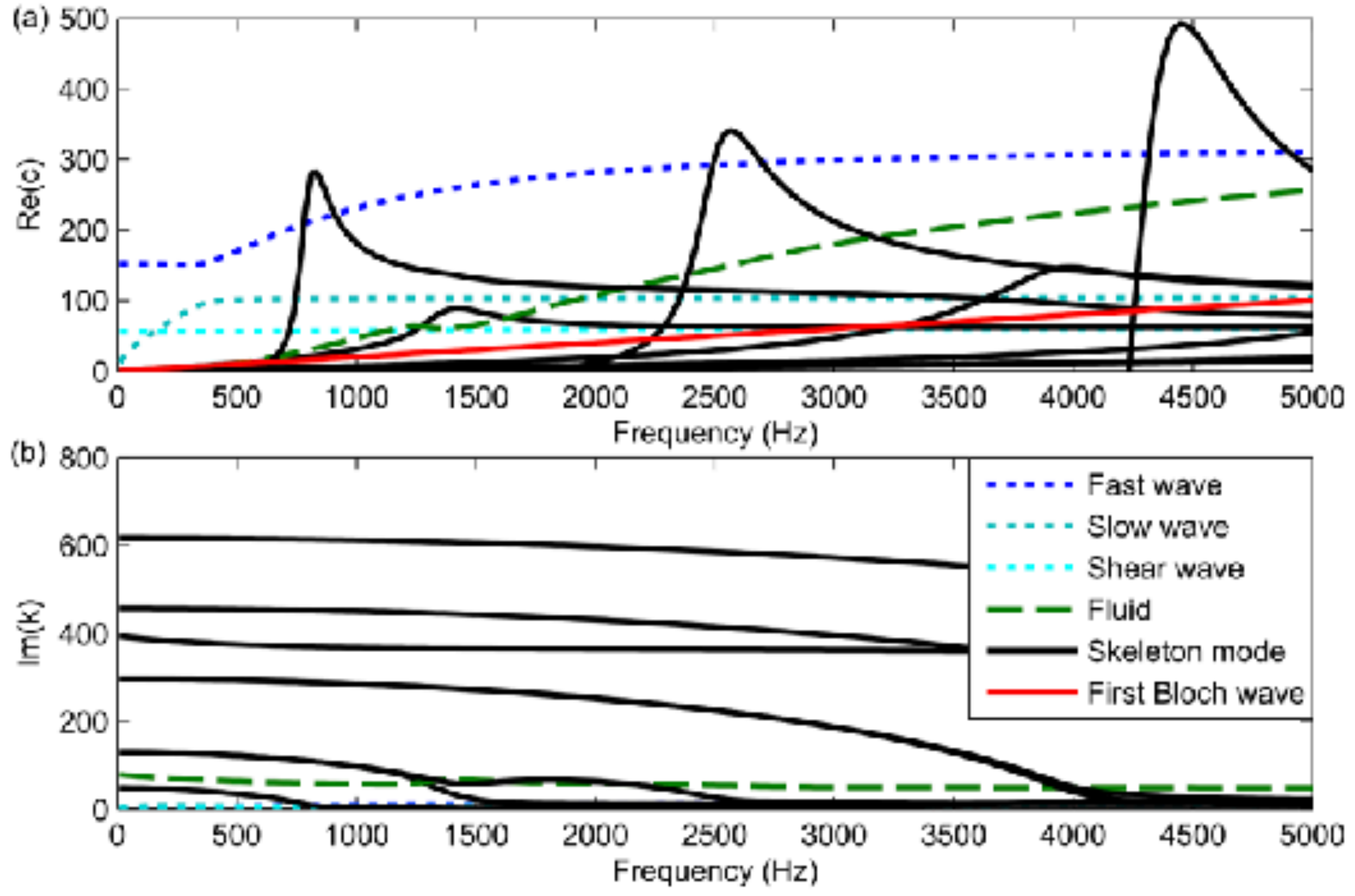
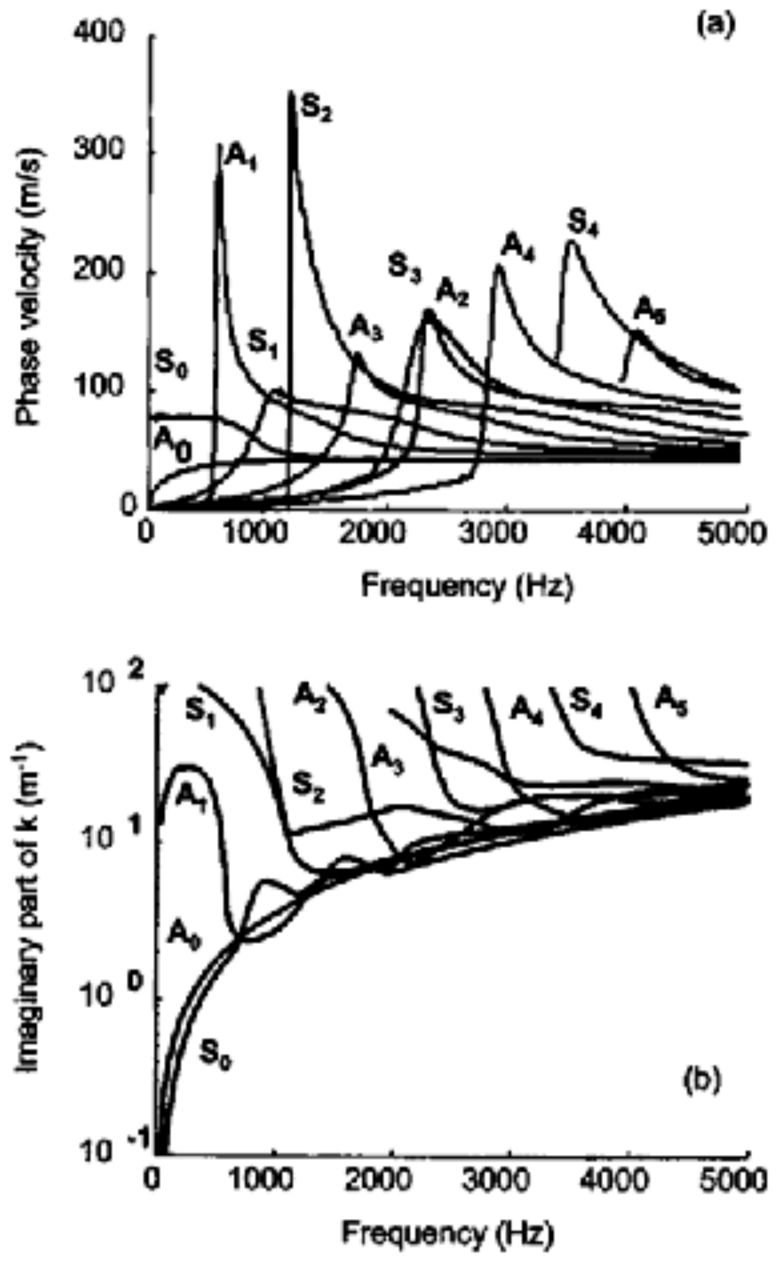
Motivations

- Scarce studies on dispersion analysis of poroelastic media in acoustics
- Computation of the dispersion diagrams of poroelastic based structures
- Existing methods: mostly root-finding



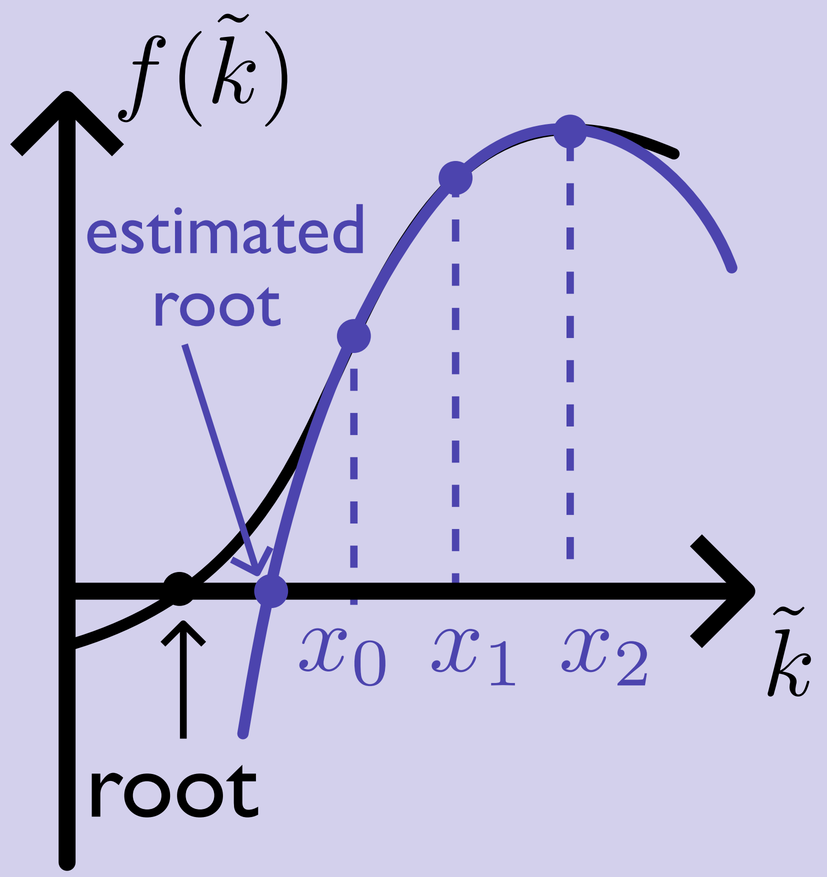
[T. Weisser *et al.*, J. Acoust. Soc. Am. **139** (2), (2016)]



[L. Boeckx *et al.*, J. Appl. Phys., **97**, 094911 (2005)]

Müller method

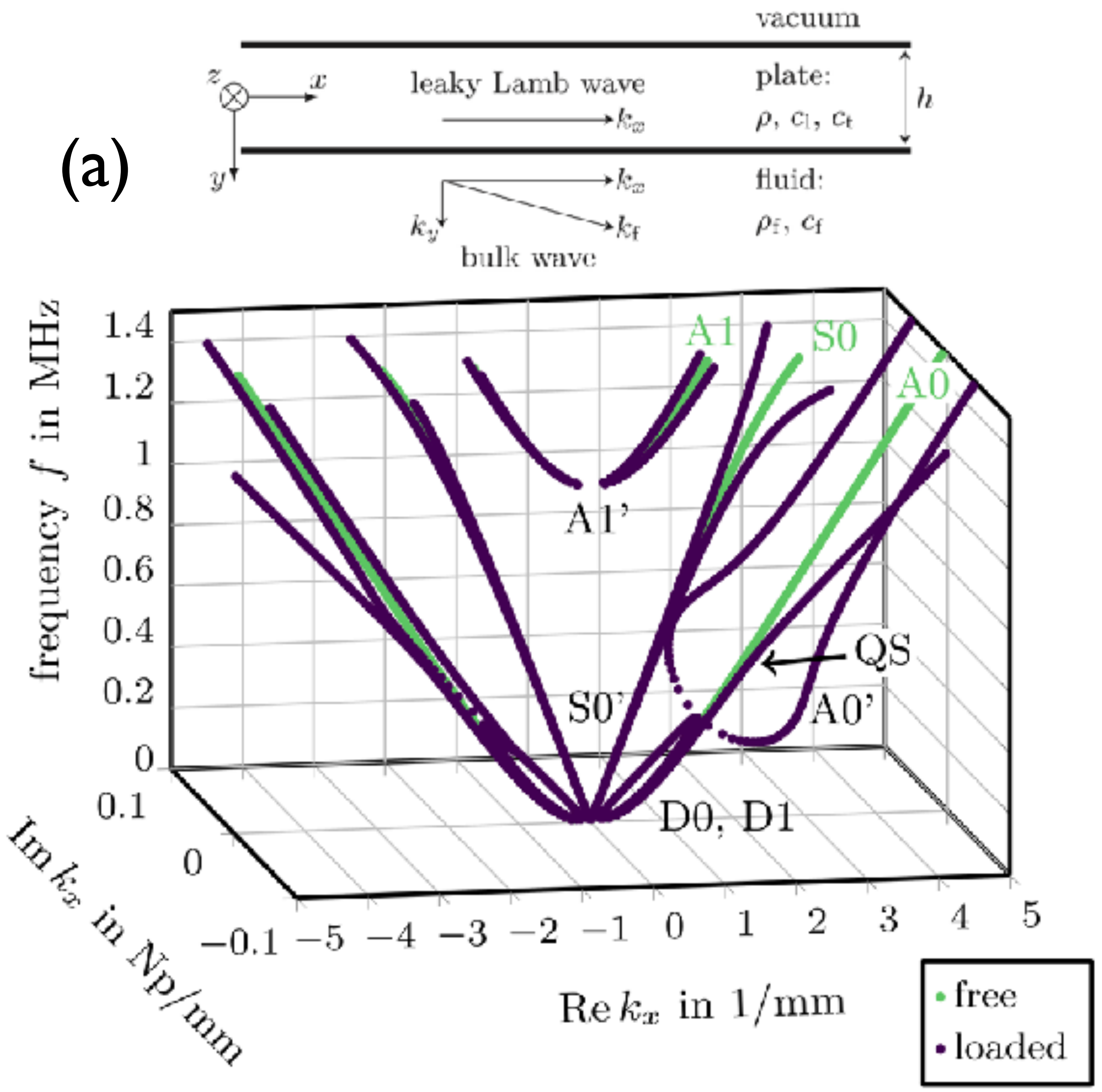
- Secant-based method
- Complex roots



Overview of spectral methods

- Spectral methods: class of numerical methods
- SCM is used widely for dispersion problems involving **elastic materials**:
  - Lamb waves [V. Pagneux *et al.*, *J. Acoust. Soc. Am.* **110**, 1307–1314 (2001)]
  - Anisotropic multilayer elastic media [F. Quintanilla *et al.*, *J Acoust Soc Am* **137**, 1180–1194 (2015)]
  - Fluid-coupled elastic layer:
    - Mapped discretized surrounding fluid [E. Georgiades *et al.*, *J. Acoust. Soc. Am.* **152**, 1487–1497 (2022)]
    - Change of variable (a) and solving a generalized eigenvalue problem (GEP) [D. Kiefer *et al.*, *J. Acoust. Soc. Am.*, **145**, 3341 (2019)]

(a)



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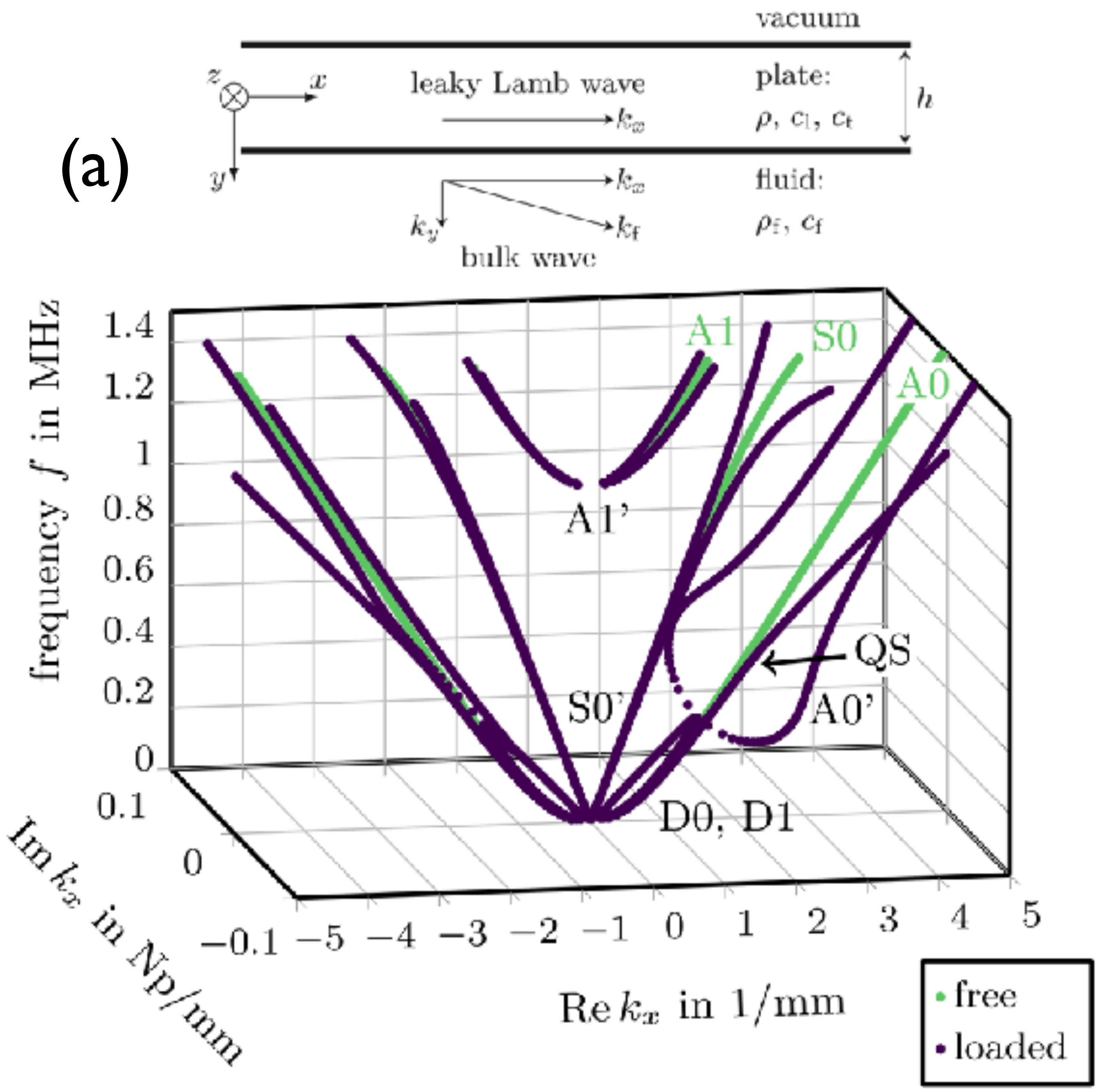
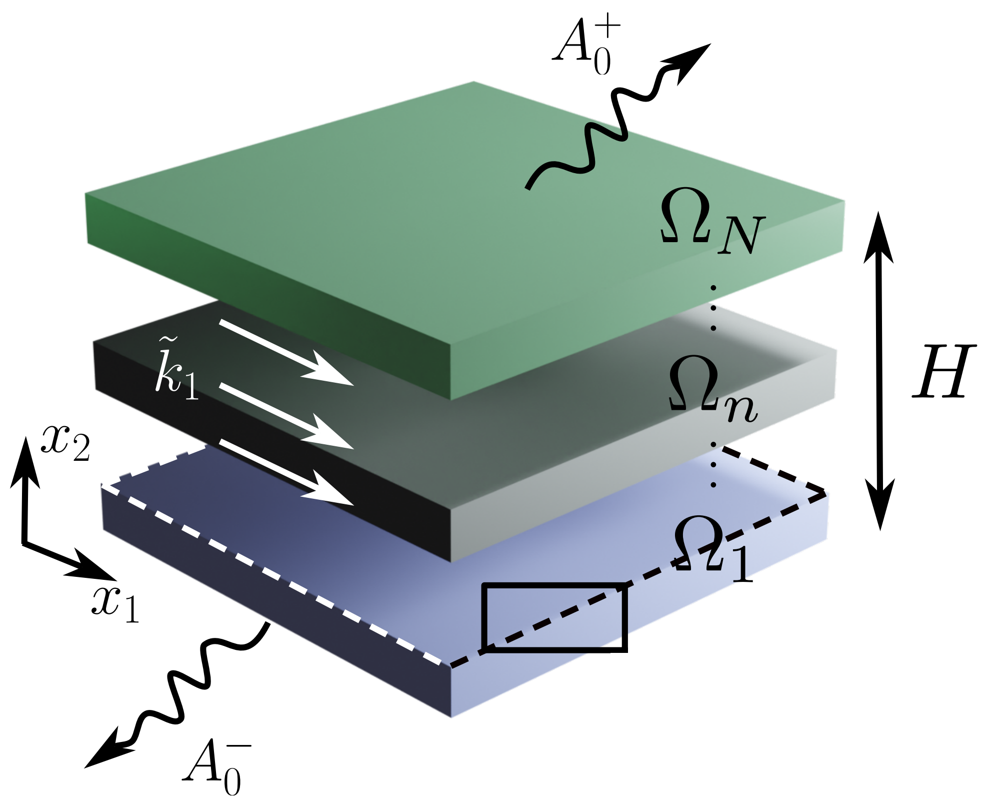


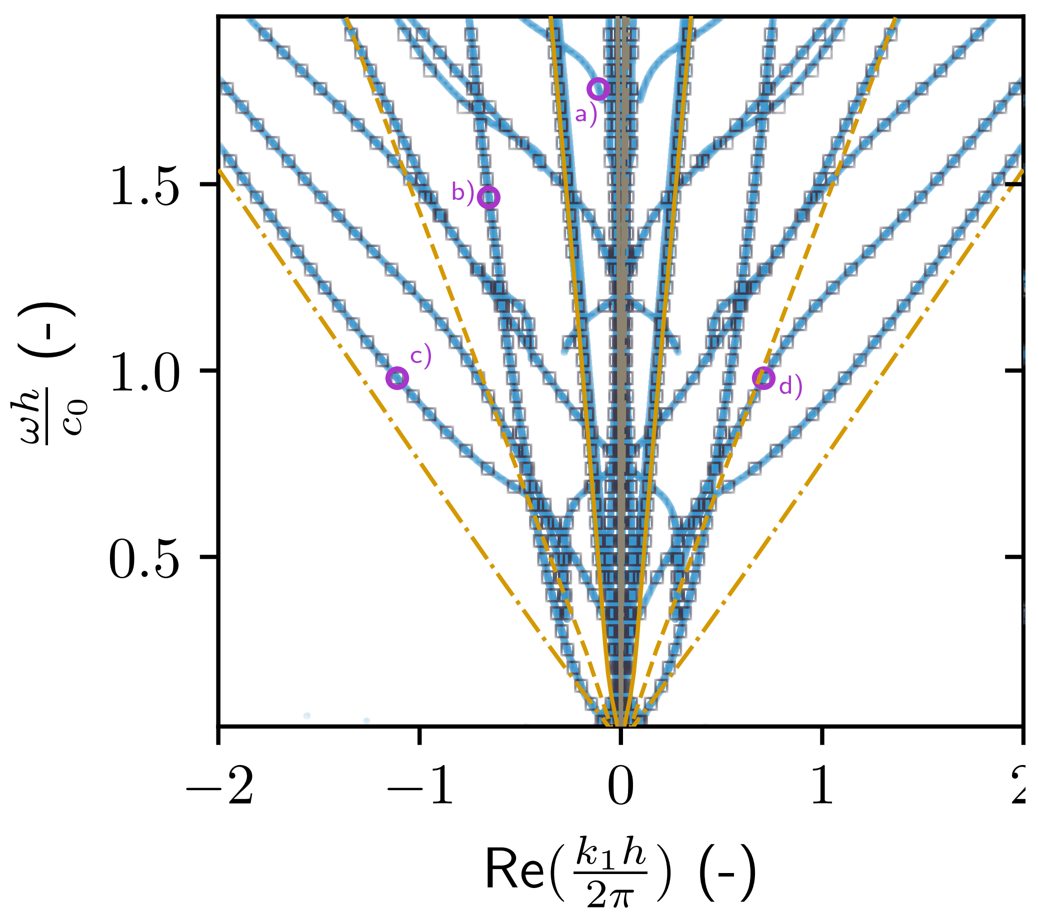
Outline

- Method description
  - Spectral collocation
  - Multilayer structure
  - Solution of the eigenvalue problem

- Results for a bilayer structure
  - Dispersion diagram & mode shapes
  - Experimental comparisons

- Conclusion & Outlooks





Spectral collocation method: generalities

- Set of fields $\mathbf{s} = \mathbf{s}(x_2)e^{ik_1x_1}e^{-i\omega t}$
- Expansion of the fields in polynomial form $\mathbf{s}(x_2) \approx \sum_{n=0}^{\infty} \alpha_n \psi_n$ (Basis functions: ψ_n)
- Chebyshev nodes distribution
- 1D Grid along an axis $x_m = \cos\left(\frac{(2m-1)\pi}{2M}\right), m = 0\ldots M$

Residue equation for each node

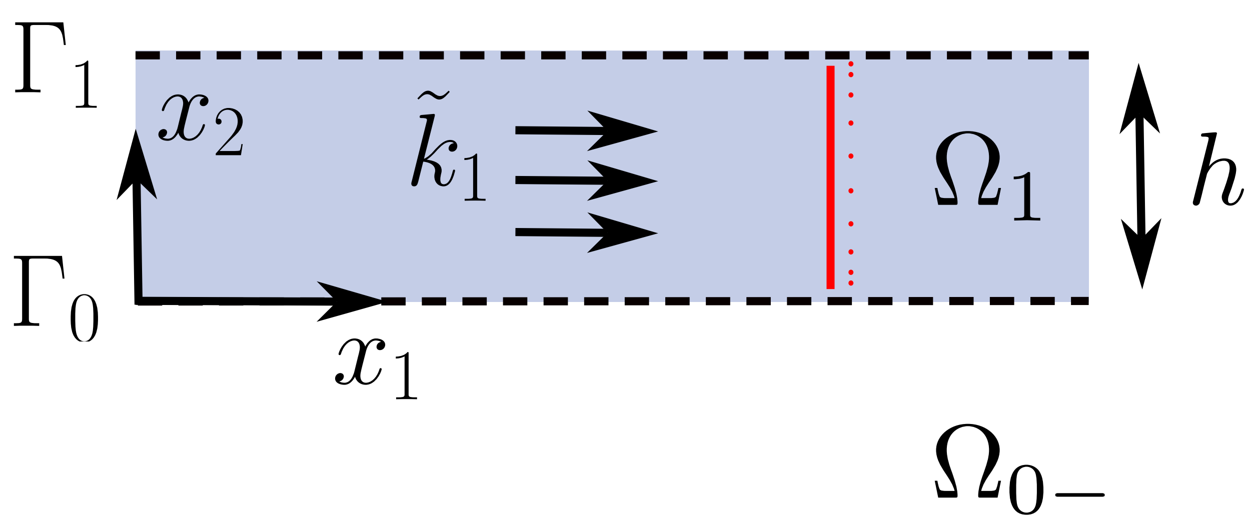
$$\sum_{n=1}^{N-1} \alpha_j \mathscr{L}[T_j](x_m) = 0$$

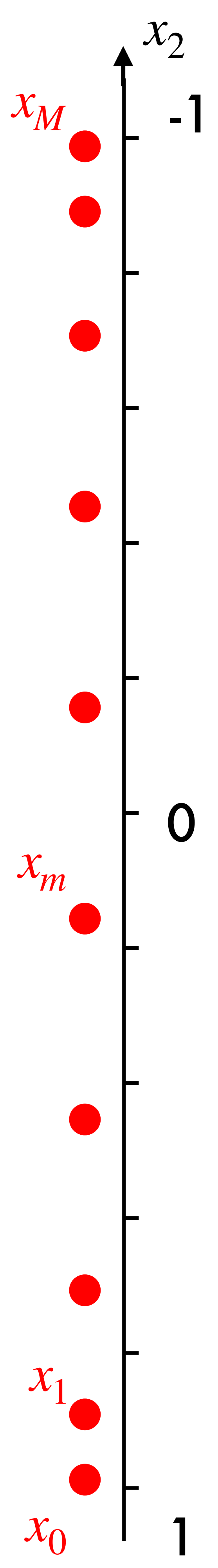
Chebyshev polynomials: T_j

- Notation for the derivatives of Chebyshev polynomials

$$\sum_n \alpha T'(x_m) \rightarrow \mathbf{T}_2$$

$$\sum_n \alpha T''(x_m) \rightarrow \mathbf{T}_{22}$$





- Linear operator $\mathscr{L}$ of the motion equations acting on a layer
- The boundary conditions Γ_0, Γ_1 are applied on the edges x_0, x_M of the grid
- Introduction of differentiation matrices

$$\frac{\partial s}{\partial x_2} \approx \left(\frac{2}{h}\right) \mathbf{D}_2^M \mathbf{s},$$

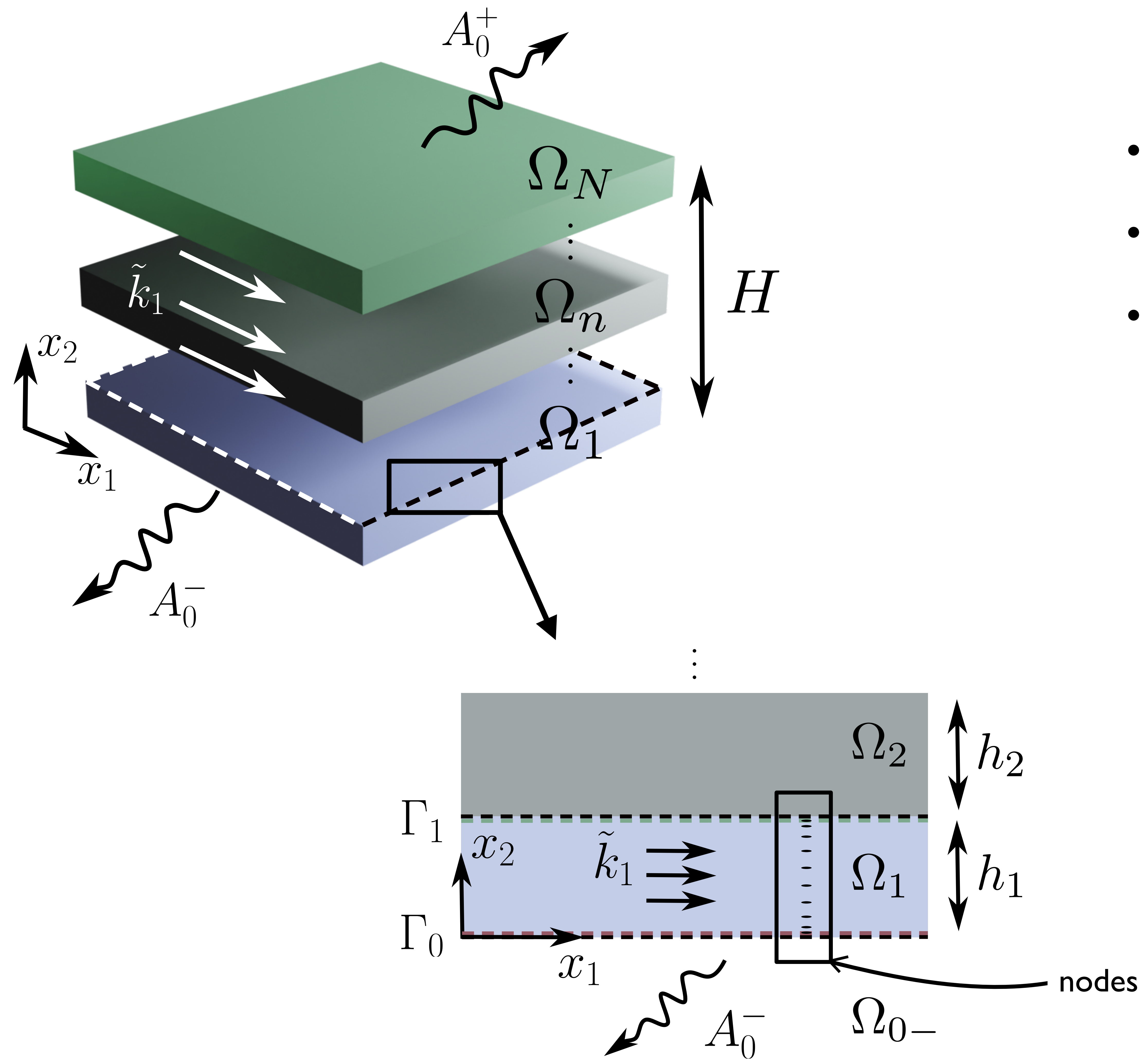
$$\frac{\partial^2 s}{\partial x_2^2} \approx \left(\frac{2}{h}\right)^2 \mathbf{D}_{22}^M \mathbf{s},$$

$$\mathbf{s} = (s_0, \ldots s_M)$$

[J.A. Weideman, S.C. Reddy, ACM Trans Math Softw, 26, **4**, 465–519 (2000)]

[L. Trefethen, *Spectral methods in MATLAB*, SIAM (2000)]

Numerical scheme for the multilayer structure



- Discretized boundary conditions $\mathbf{\Gamma}^- s_n + \mathbf{\Gamma}^+ s_{n+1} = 0$
- Set of motion equations for each layer at $\boldsymbol{\alpha}_n$
- Radiating amplitude in an infinite surrounding fluid $A_0^\pm$

$$\mathbf{K}\,\mathbf{S} = \begin{pmatrix} \mathbf{\Gamma}_0^- & \mathbf{\Gamma}_0^+ & 0 & & & \\ 0 & \boldsymbol{\alpha}_1 & 0 & & & \\ 0 & \mathbf{\Gamma}_1^- & \mathbf{\Gamma}_1^+ & 0 & & \\ & 0 & \boldsymbol{\alpha}_2 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & \boldsymbol{\alpha}_N & 0 \\ & & & 0 & \mathbf{\Gamma}_N^- & \mathbf{\Gamma}_N^+ \end{pmatrix} \begin{pmatrix} A_{0-} \\ \mathbf{s}^{(1)} \\ \mathbf{s}^{(2)} \\ \vdots \\ \mathbf{s}^{(N)} \\ A_{0+} \end{pmatrix} = \mathbf{0}\,.$$

Case of the SCM with a poroelastic layer

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Case of the SCM with a poroelastic layer

- Discretization of the (u_s, p) formulation equations

$$\nabla \cdot \hat{\boldsymbol{\sigma}}^s + \tilde{\rho}\omega^2 \mathbf{u}^s + \tilde{\gamma}\,\nabla p = 0,$$
$$\frac{\nabla^2 p}{\tilde{\rho}_{22}\omega^2} - \frac{\tilde{\gamma}}{\phi^2}\nabla \cdot \boldsymbol{u}^s + \frac{1}{\tilde{R}}p = 0.$$

[N. Atalla *et al.*, J. Acoust. Soc. Am. **104**, 3 (1998)]

- 3 set of unknowns instead of 4 with the (u_s, u_f) formulation

- Coupling at the interfaces [P. Debergue *et al.*, *J. Acoust. Soc. Am.* **106**, 2383–2390 (1999)]
  - Fluid-PEM interfaces in the interface Γ_0
  - Γ_1, relation of the fields u_1^s, u_2^s, p with the fields s_2 in $\boldsymbol{\alpha}_2$ using the appropriate interface conditions
- Notation used

$$\hat{P} = \hat{A} + 2N_s; \quad \hat{A} = \tilde{A} - \frac{\tilde{Q}^2}{\tilde{R}}; \quad \tilde{\rho} = \rho_{11} - \frac{\tilde{\rho}_{12}^2}{\tilde{\rho}_{22}};$$
$$\tilde{\gamma} = \phi\left(\frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} - \frac{\tilde{Q}}{\tilde{R}}\right); \quad \hat{\beta} = 1 + \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}}$$

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$$\boldsymbol{\alpha}_1 = \begin{pmatrix} N_s\boldsymbol{T}_{22} + (\omega^2\tilde{\rho} - \hat{P}k_1^2)\boldsymbol{T} & \left(\hat{P} - N_s\right) ik_1\boldsymbol{T}_2 & i\tilde{\gamma}k_1\boldsymbol{T} \\ \left(\hat{P} - N_s\right) ik_1\boldsymbol{T}_2 & \hat{P}\boldsymbol{T}_{22} + (\omega^2\tilde{\rho} - N_s k_1^2)\boldsymbol{T} & \tilde{\gamma}\boldsymbol{T}_2 \\ -ik_1\tilde{\gamma}/\phi^2\boldsymbol{T} & -\tilde{\gamma}/\phi^2\boldsymbol{T}_2 & \boldsymbol{T}_{22}/\tilde{\rho}_{22}\omega^2 - \left(k_1^2/\tilde{\rho}_{22}\omega^2 + \frac{1}{\tilde{R}}\right)\boldsymbol{T} \end{pmatrix}$$

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Linearization of the generalized eigenvalue problem

The matrix system previously obtained is

$$\mathbf{KS} = \mathbf{0}, \quad \text{with} \quad \mathbf{S} = (A_{0-}, \mathbf{s}_0, \ldots s_n, \ldots, s_N, A_{0+})\,.$$

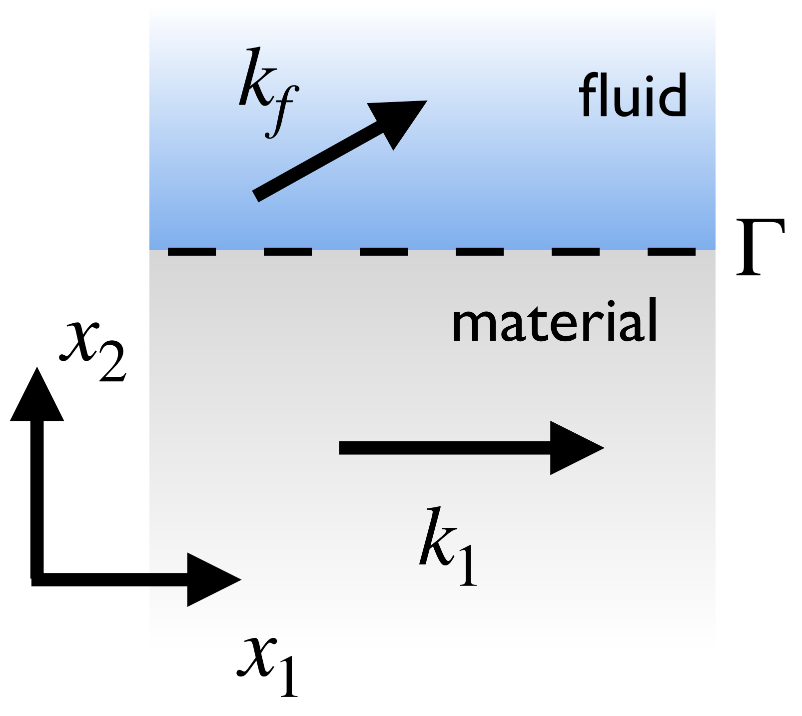
It can be rewritten as

$$\left(k_1^2\mathbf{K}_2 + k_1\mathbf{K}_1 + \mathbf{K}_0 + ik_2^{(f)}\mathbf{K}_1'\right)\mathbf{S} = \mathbf{0}$$

$$\text{with} \quad k_2^{(f)} = \sqrt{\left(\frac{\omega}{c_f}\right)^2 - k_1^2},$$

which is a non-linear GEP. Let

$$k_1 = k^{(f)}\frac{\gamma + \gamma^{-1}}{2}, \quad k_2 = k^{(f)}\frac{\gamma - \gamma^{-1}}{2i}.$$



By using companion linearization,

$$\left(\mathbf{A} - \gamma\mathbf{B}\right)\mathbf{S}' = \mathbf{0}$$

eigenvalues

eigenvectors

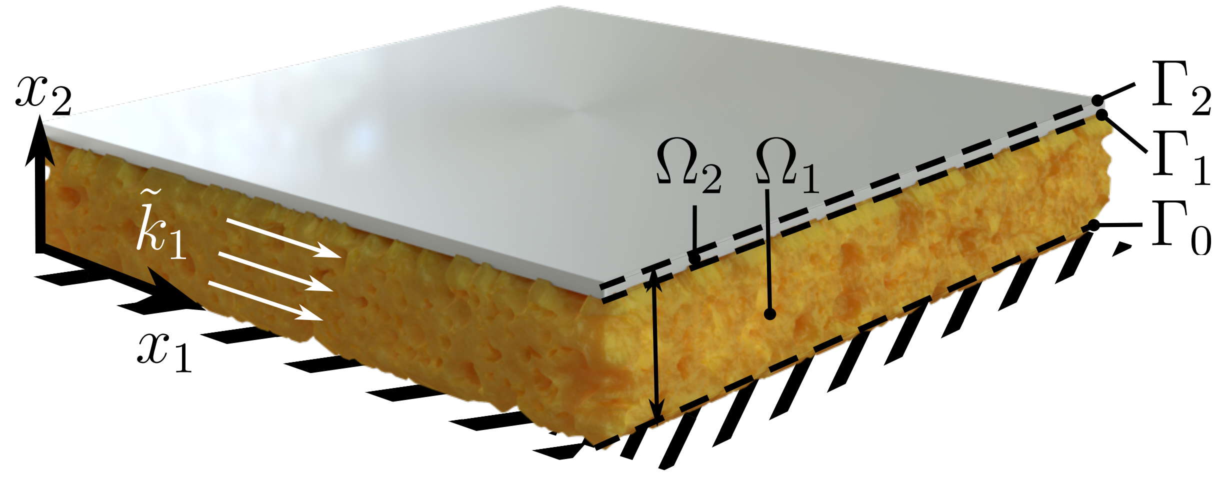
$$\mathbf{S}' = \begin{pmatrix} \gamma^3\mathbf{S} \\ \gamma^2\mathbf{S} \\ \gamma\mathbf{S} \\ \mathbf{S} \end{pmatrix}$$

GEP solvable with traditional eigenvalue solvers

[D. Kiefer *et al.*, *J. Acoust. Soc. Am.*, **145**, 3341 (2019)]

[A. Hood, Localizing the eigenvalues of matrix-valued functions: analysis and applications, Cornell University (2017)]

Dispersion relation of the structure



- Melamine foam, rigidly-backed with an aluminium plate on top
- Coupling between the elastic plate and the melamine at Γ_1

$$\begin{pmatrix} \mathbf{\Gamma}_0 & 0 & 0 \\ \boldsymbol{\alpha}_1 & 0 & 0 \\ \mathbf{\Gamma}_1^- & \mathbf{\Gamma}_1^+ & 0 \\ 0 & \boldsymbol{\alpha}_2 & 0 \\ 0 & \mathbf{\Gamma}_2^- & \Gamma_2^+ \end{pmatrix} \begin{pmatrix} \mathbf{s}^{(p)} \\ \mathbf{s}^{(e)} \\ A_{0+} \end{pmatrix} = \mathbf{0}$$

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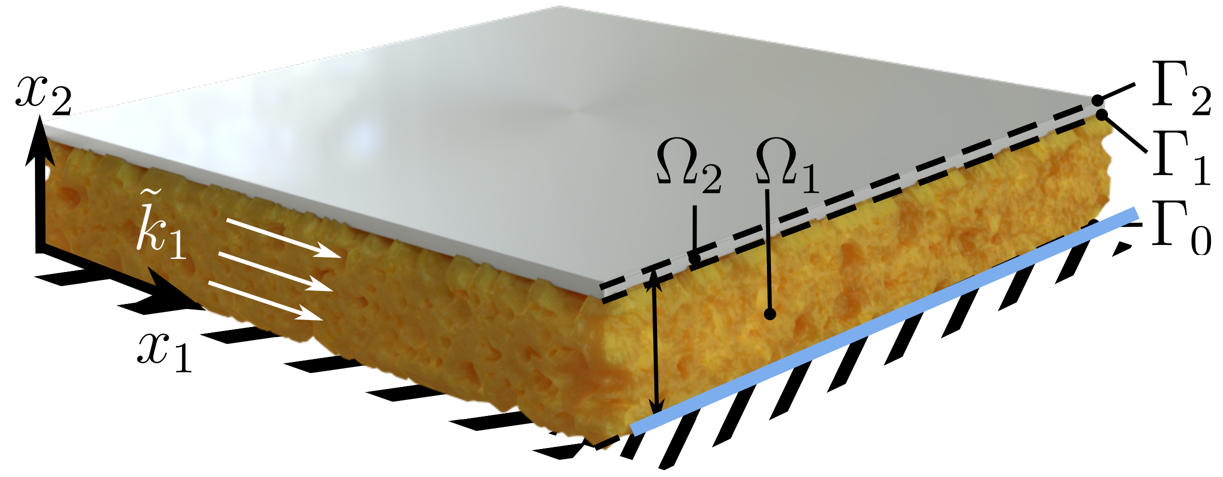

Rigid-backing BC

$$\begin{pmatrix} \Gamma_0 & 0 & 0 \\ \boldsymbol{\alpha}_1 & 0 & 0 \\ \boldsymbol{\Gamma}_1^- & \boldsymbol{\Gamma}_1^+ & 0 \\ 0 & \boldsymbol{\alpha}_2 & 0 \\ 0 & \boldsymbol{\Gamma}_2^- & \Gamma_2^+ \end{pmatrix} \begin{pmatrix} \mathbf{s}^{(p)} \\ \mathbf{s}^{(e)} \\ A_{0+} \end{pmatrix} = \mathbf{0}$$

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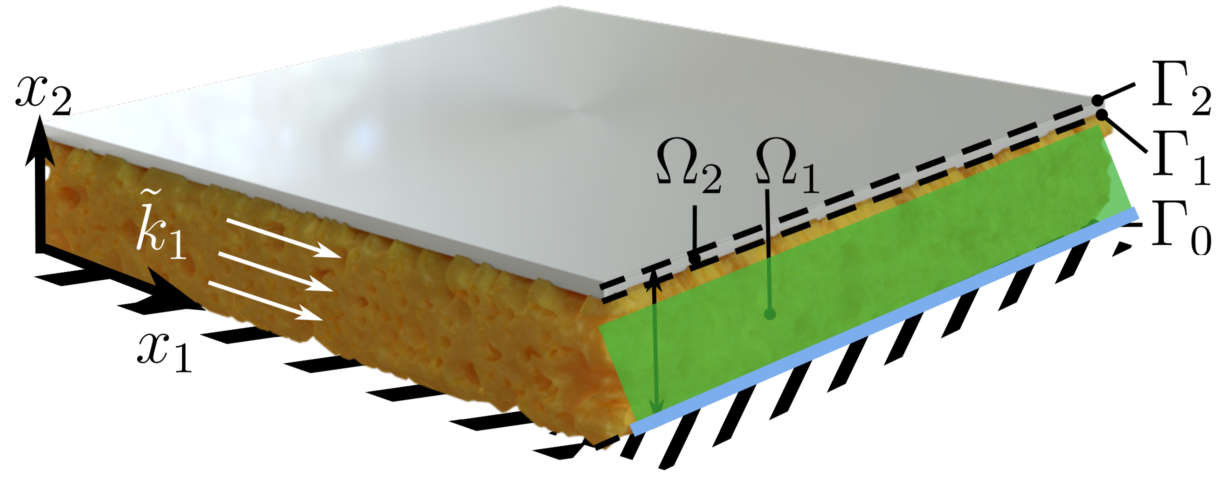

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Rigid-backing BC

Biot model

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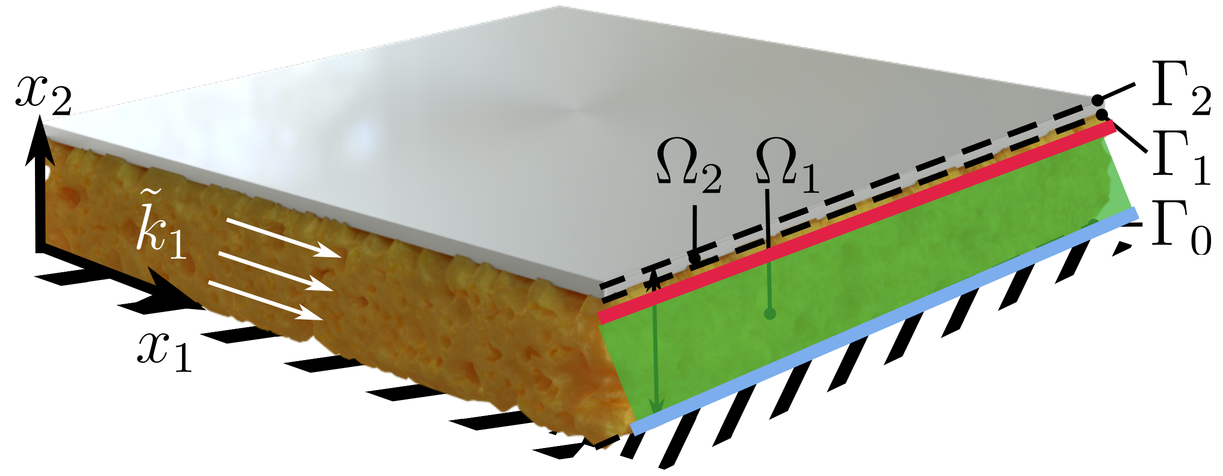

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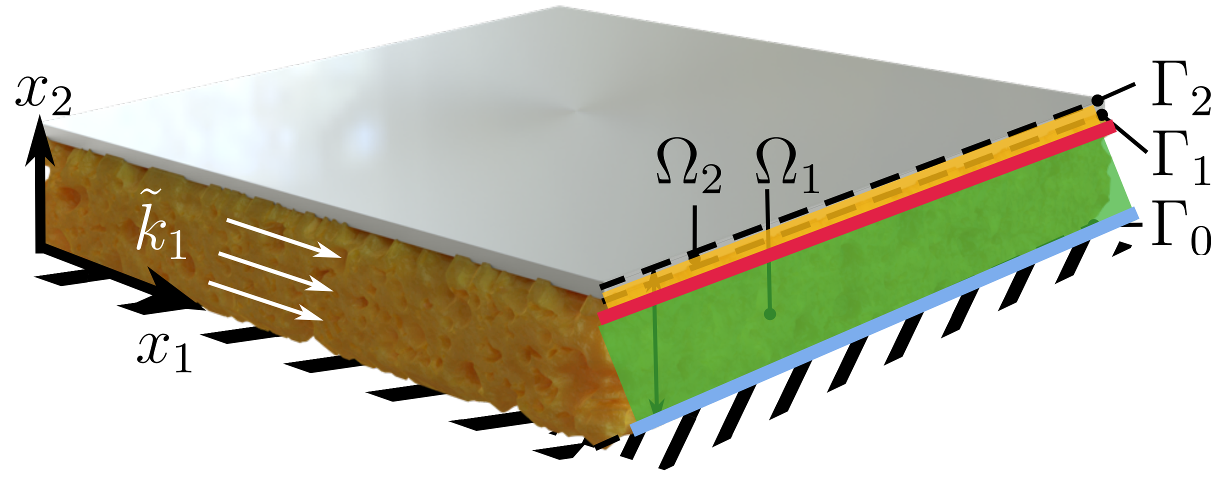

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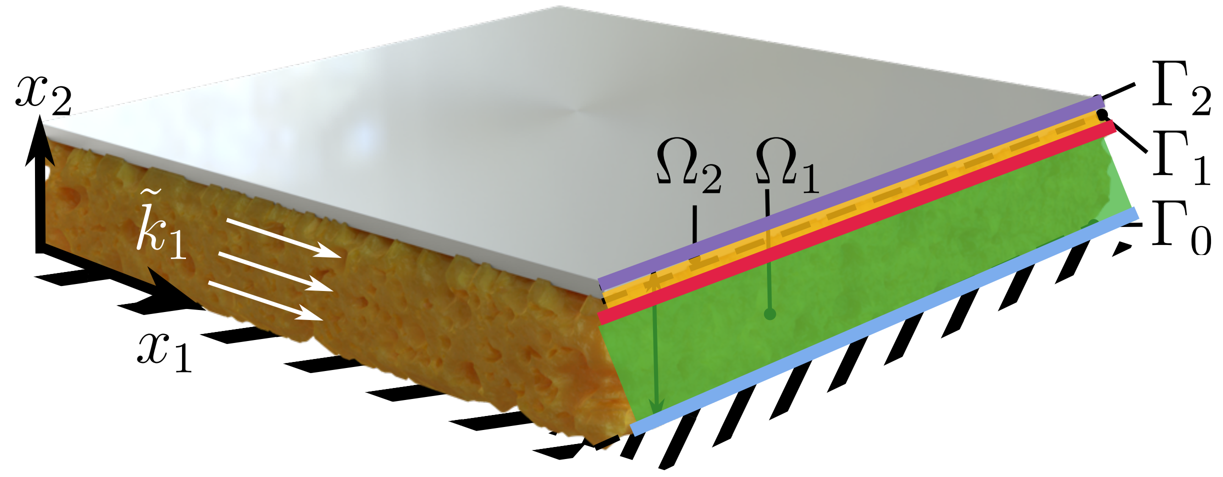

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Elastic-fluid coupling

Dispersion relation of the structure

- Melamine foam, rigidly-backed with an aluminium plate on top
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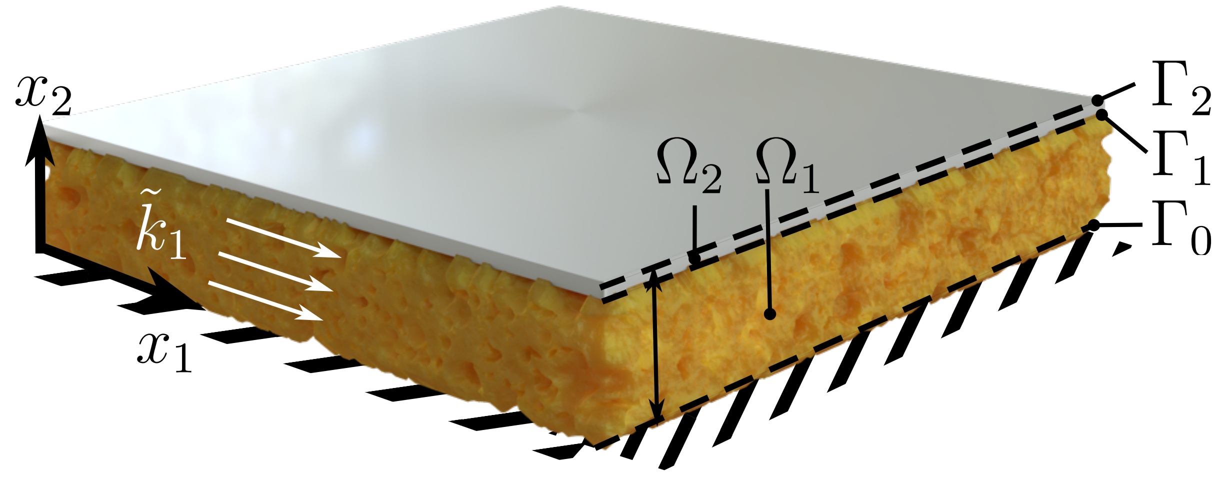
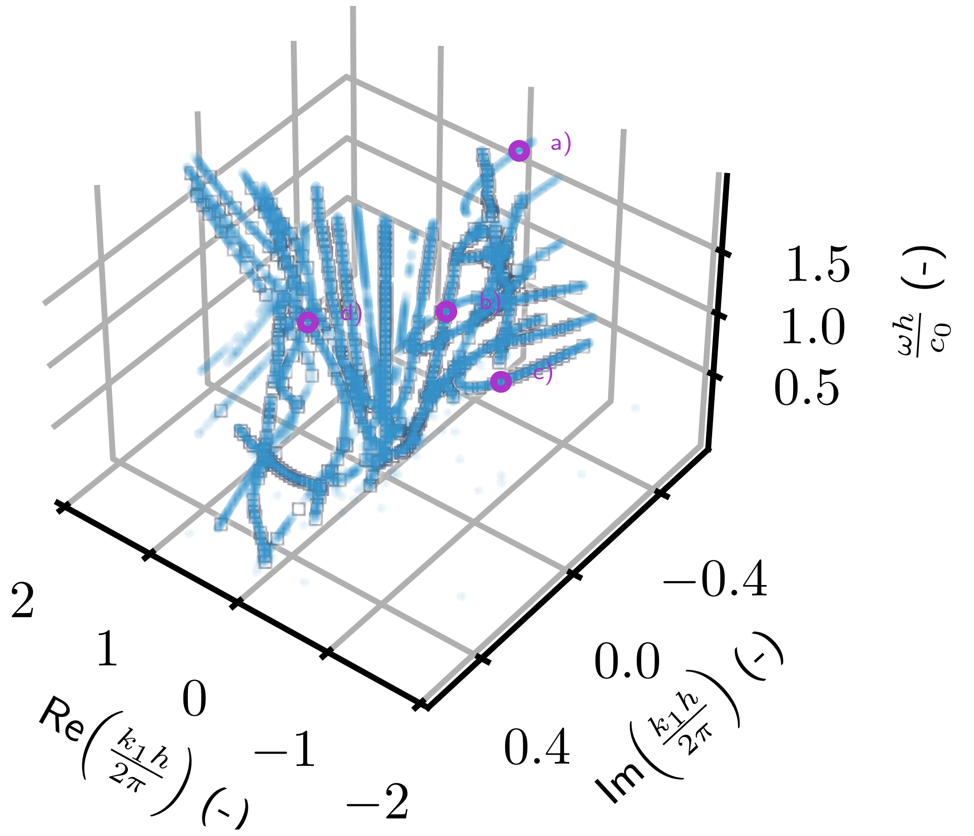
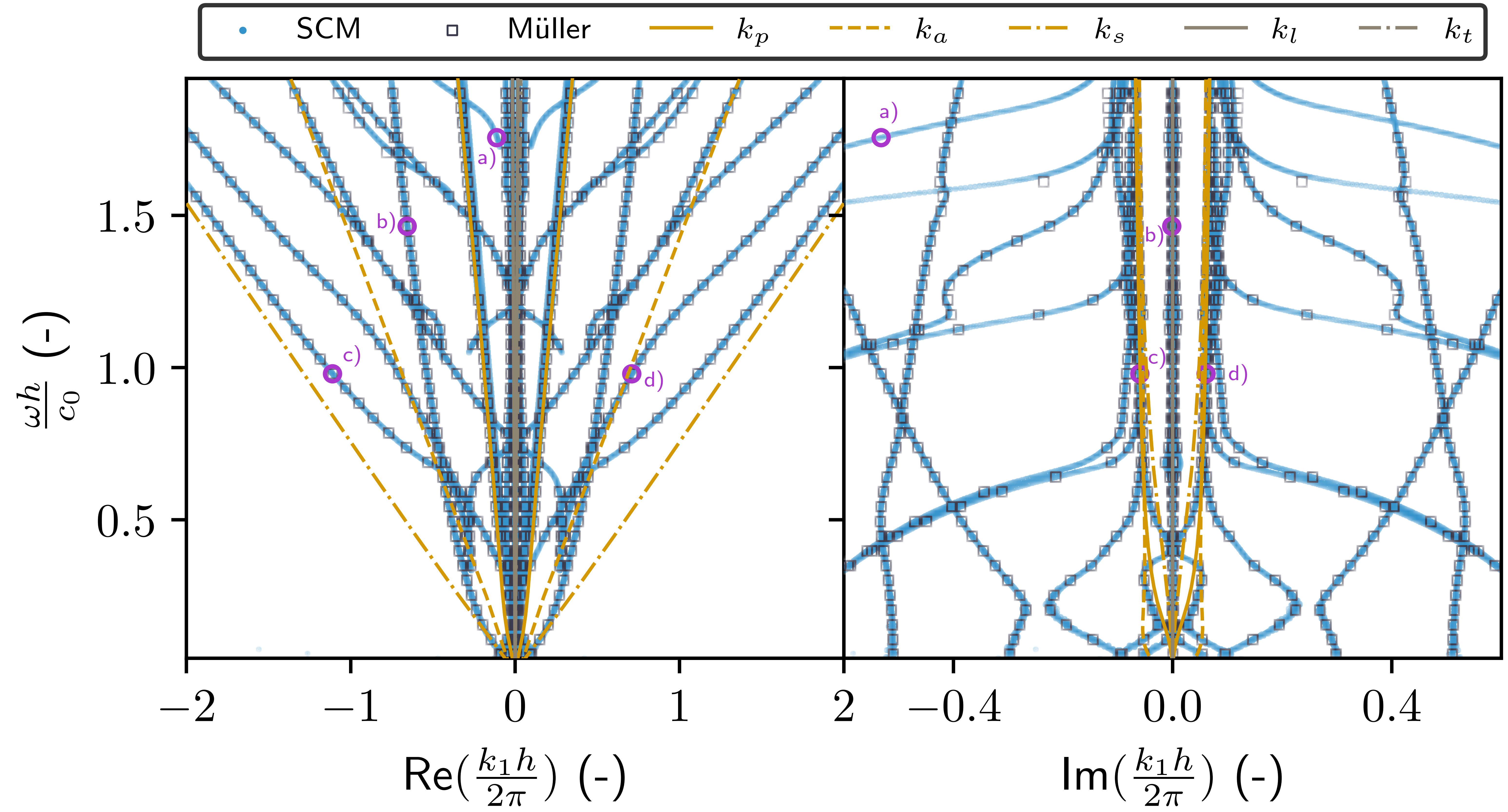


$$\begin{pmatrix} \boldsymbol{\Gamma}_0 & 0 & 0 \\ \boldsymbol{\alpha}_1 & 0 & 0 \\ \boldsymbol{\Gamma}_1^- & \boldsymbol{\Gamma}_1^+ & 0 \\ 0 & \boldsymbol{\alpha}_2 & 0 \\ 0 & \boldsymbol{\Gamma}_2^- & \Gamma_2^+ \end{pmatrix} \begin{pmatrix} \mathbf{s}^{(p)} \\ \mathbf{s}^{(e)} \\ A_{0+} \end{pmatrix} = \mathbf{0}$$





Retrieving the mode shapes of the system

- From $\left(\mathbf{A} - \gamma \mathbf{B}\right) \mathbf{S}' = \mathbf{0}$, for a given $\tilde{k}_1 \to s(x_j)$
  - Expansion coefficients $\alpha[s] = \boldsymbol{T}_m^{-1}(x_j) s(x_j)$
  - Reinterpolated fields $s(x_2) = \alpha[s] T_m(x_2)$
- u_1^s, u_2^s, p for the PEM, and u_1^s, u_2^s for the elastic layer
- Fluid displacement computed using $\boldsymbol{u}_f = \dfrac{\phi}{\tilde{\rho}_{22}\omega^2} \nabla p - \dfrac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} \boldsymbol{u}_s$
- Calculation of the Umov-Poynting vectors in the PEM $\boldsymbol{P}_t = -\dfrac{1}{2} \boldsymbol{\sigma} \boldsymbol{v}^\star$

[J. M. Carcione, Wave fields in real media, Elsevier (2007)]

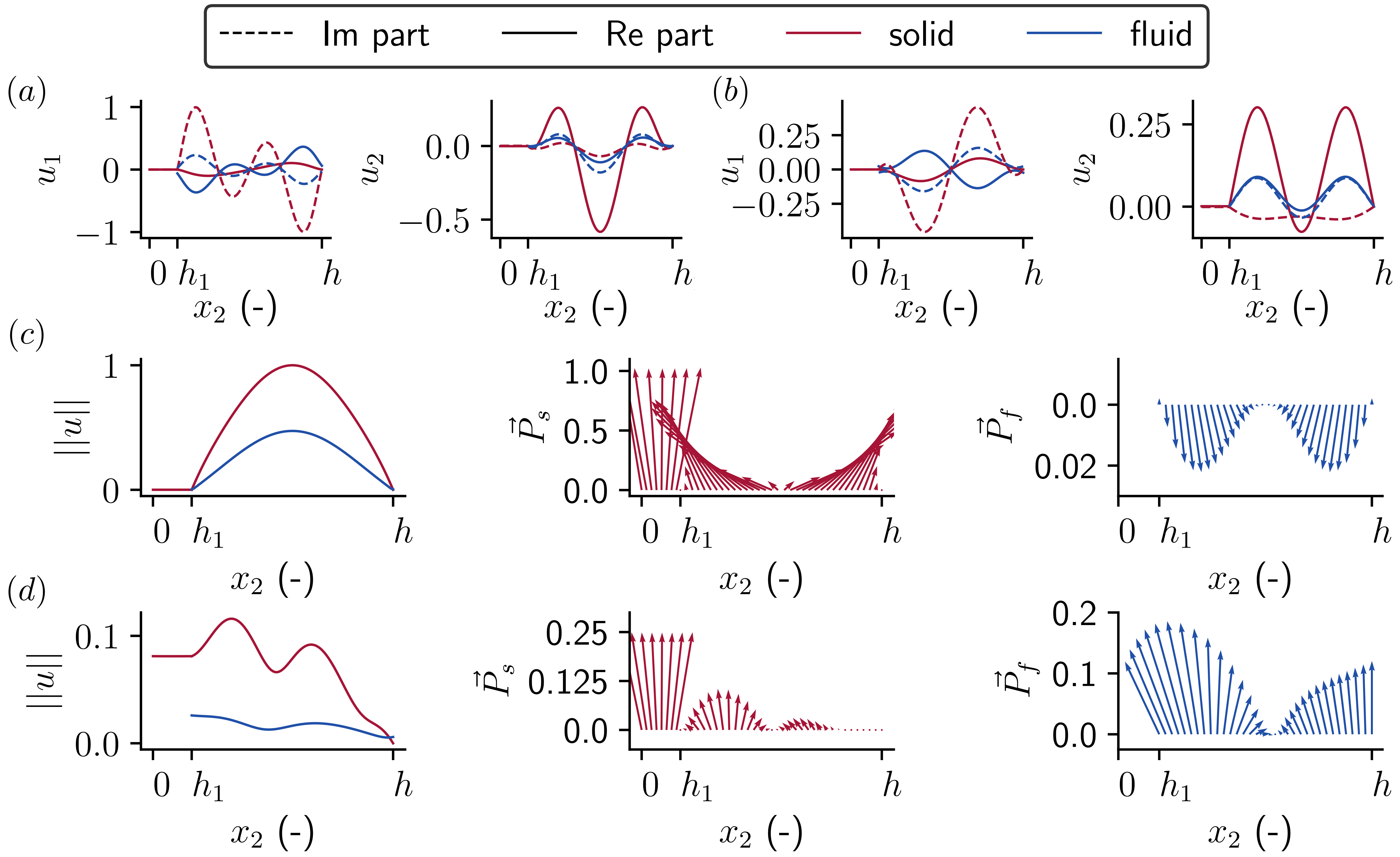
Decoupled solid and fluid phases

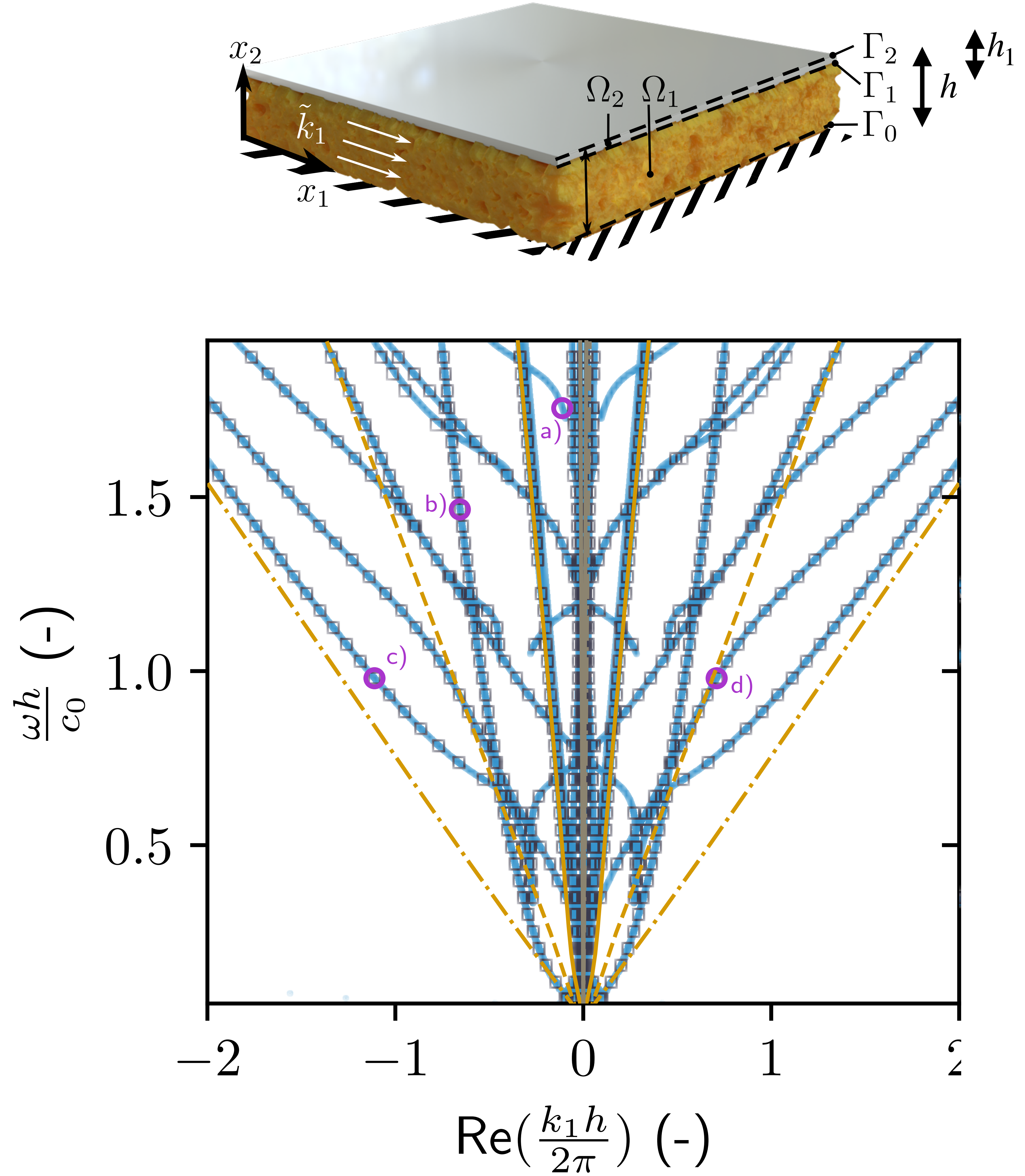
$$\begin{cases} P_s = -\dfrac{1}{2} \boldsymbol{\sigma}_s \boldsymbol{v}_s^\star \\ P_f = -\dfrac{1}{2} \boldsymbol{\sigma}_f \boldsymbol{v}_f \end{cases}$$

$$\boldsymbol{v}_s = -i\omega \boldsymbol{u}_s \qquad \boldsymbol{\sigma}_s = \underbrace{\hat{A} \nabla \cdot \boldsymbol{u}_s \boldsymbol{I} + 2\tilde{N}_s \boldsymbol{\epsilon}_s}_{\hat{\boldsymbol{\sigma}}_s} - \phi \frac{\tilde{Q}}{\tilde{R}} p \boldsymbol{I}$$

$$\boldsymbol{v}_f = -i\omega \boldsymbol{u}_f \qquad \boldsymbol{\sigma}_f = -\phi p \boldsymbol{I}$$

Mode shapes & Umov-Poynting vectors

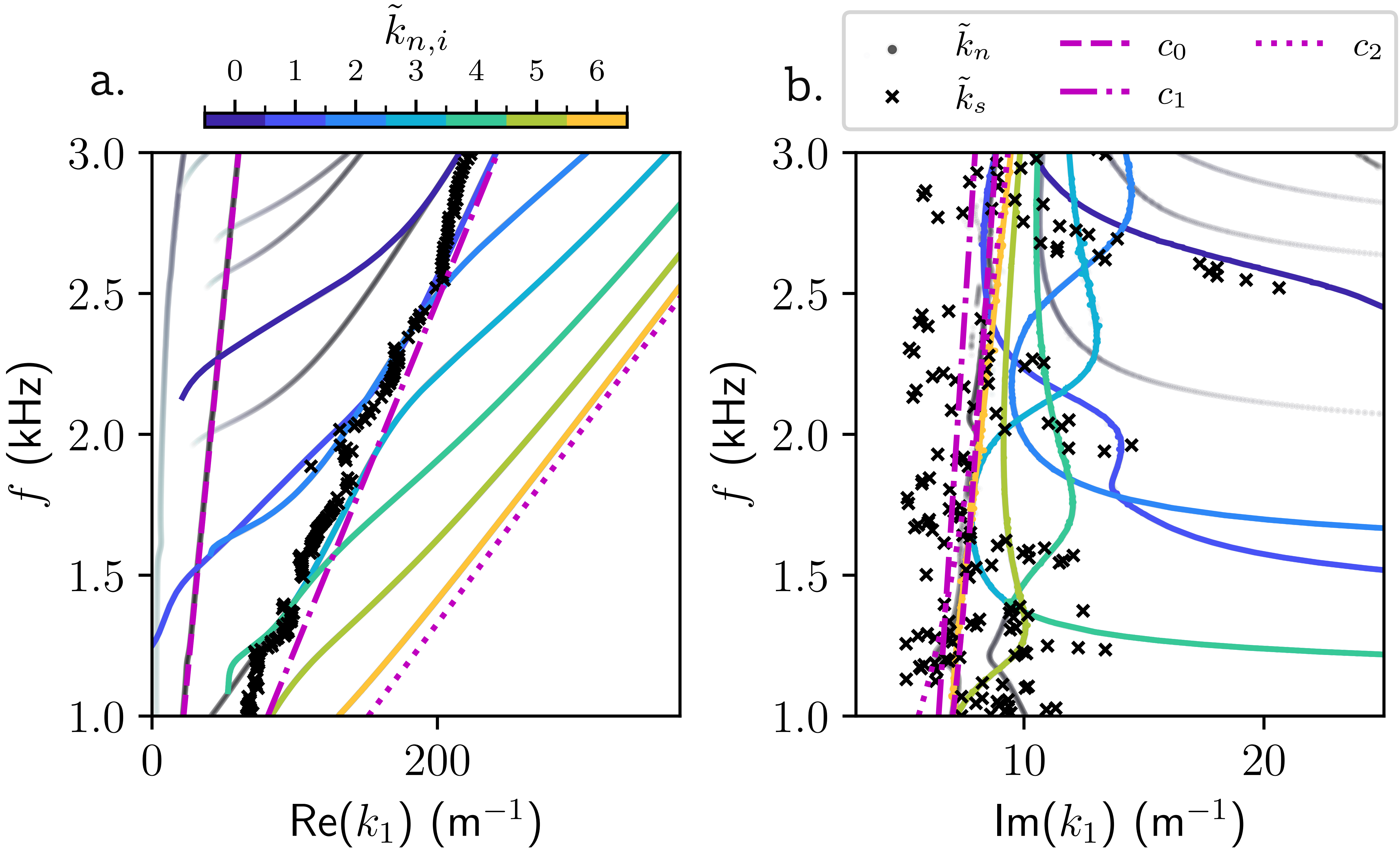


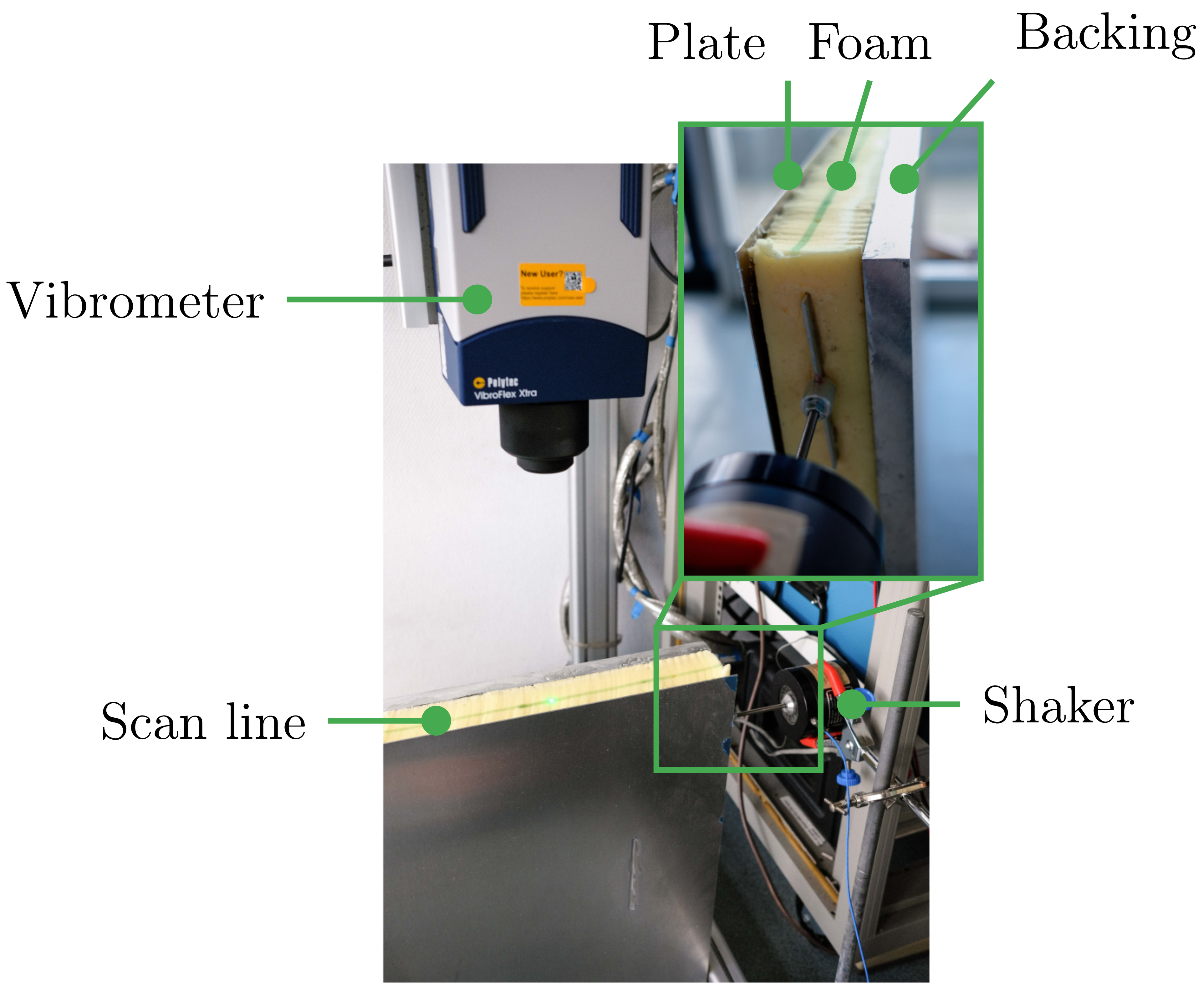


Experimental comparisons

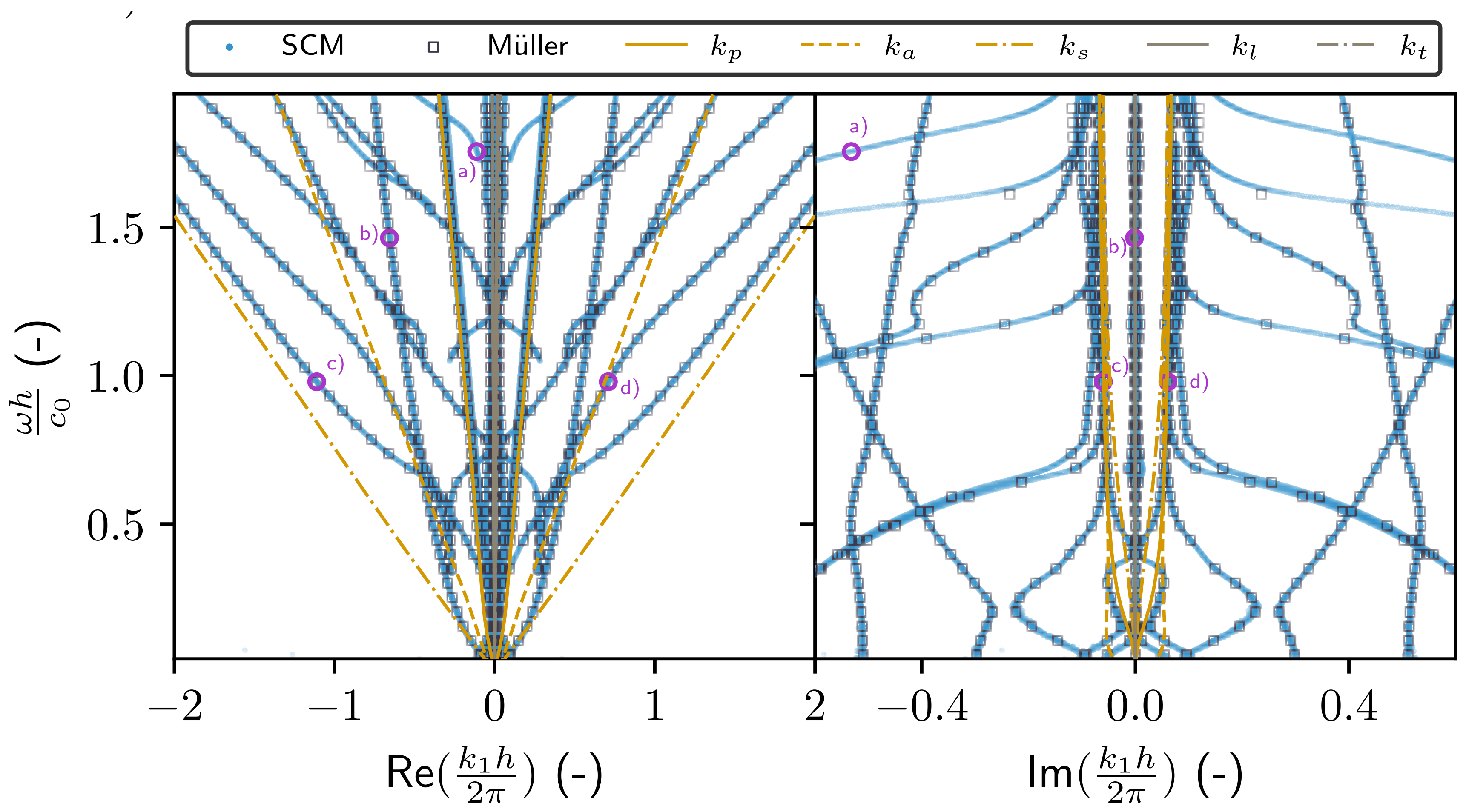
- Experimental configuration, with the same bilayer structure
- Scan measurement of the normal displacement on the sample
- Recovery of complex wavenumbers $\tilde{k}_s$ using the SLaTCoW method

[A. Geslain *et al.* J. Appl. Phys. **120**, 13, p.135107 (2016)]



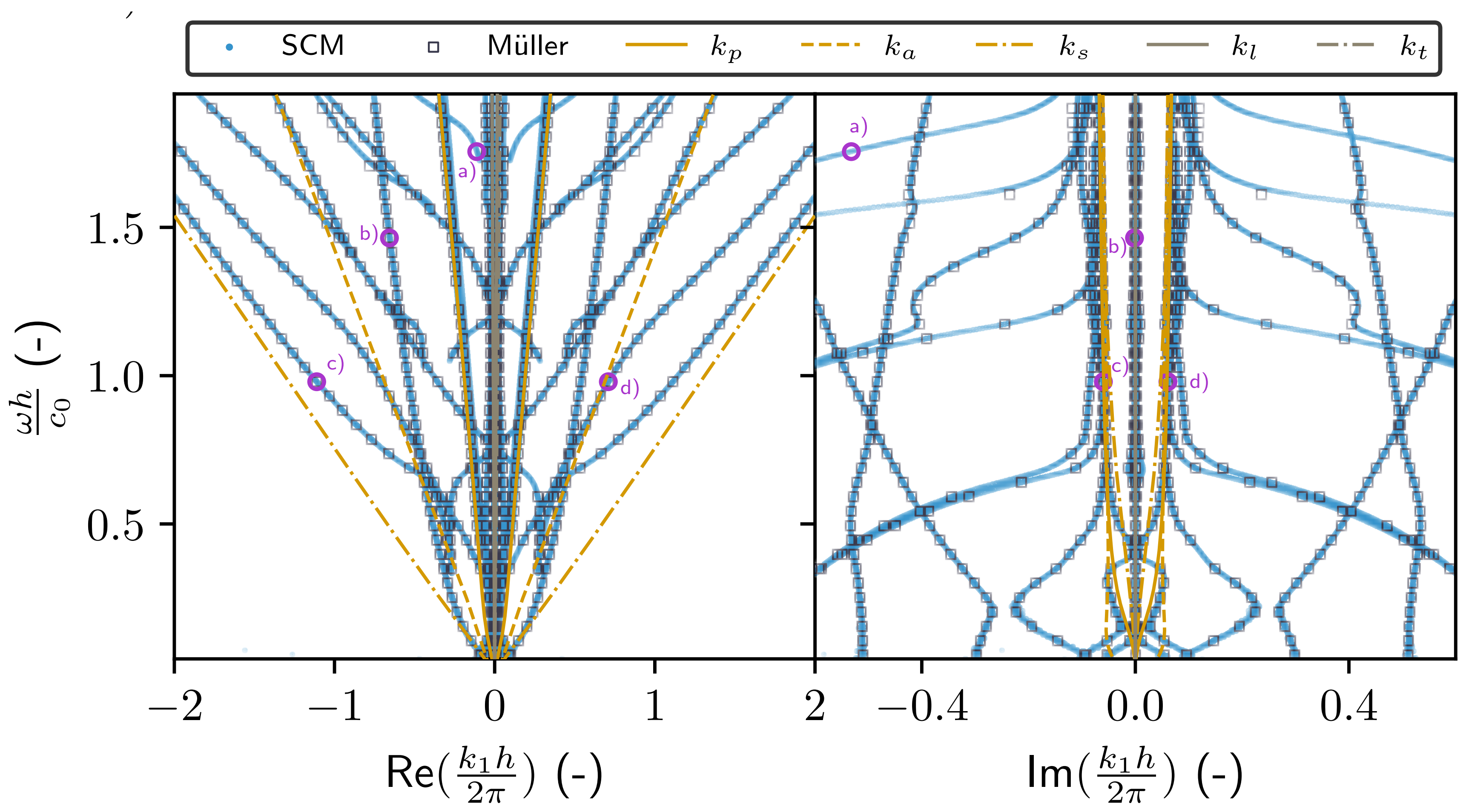


Conclusion & Outlooks



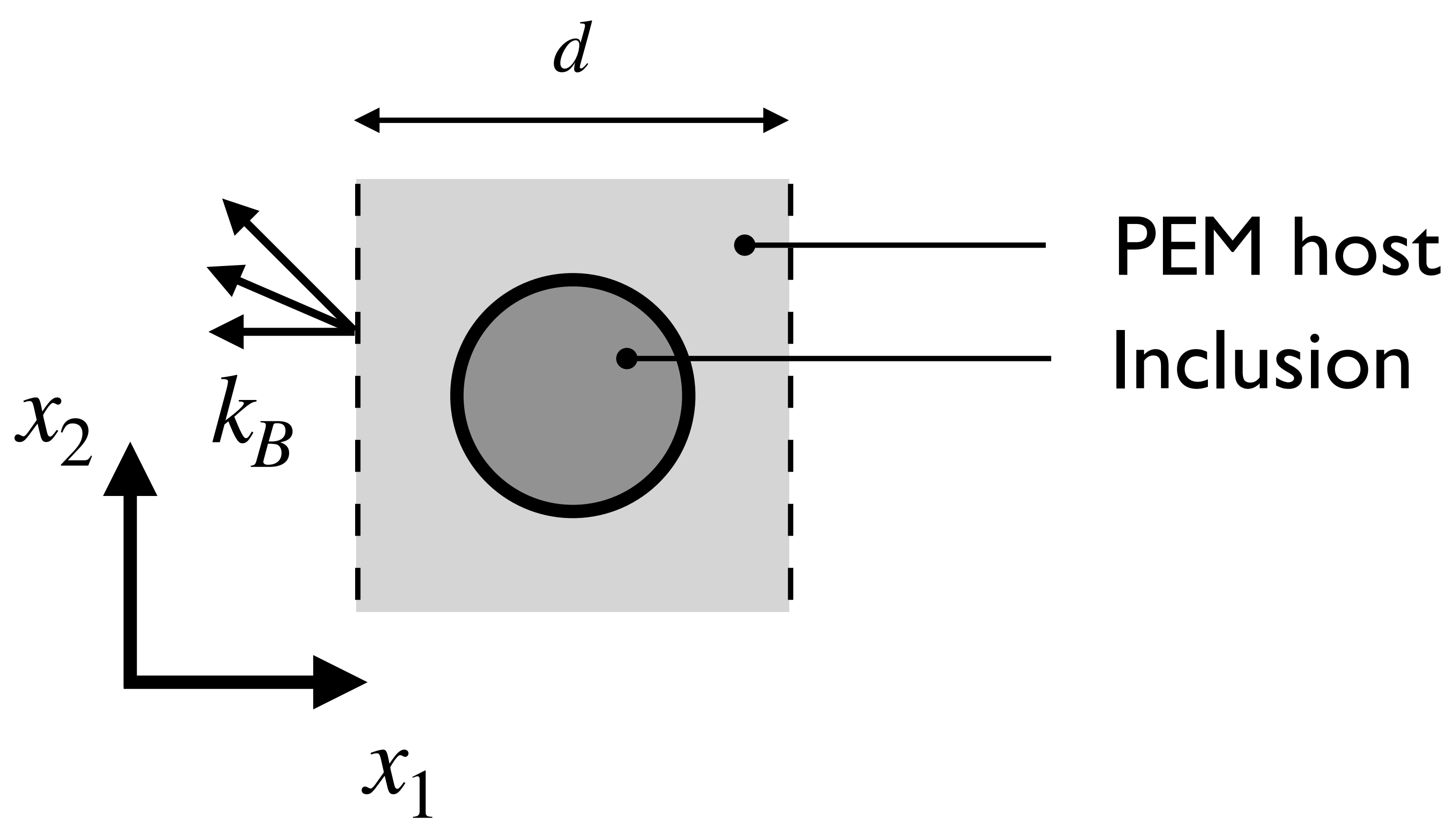
- Numerical method to study guided waves in general multilayer structures
- Fast and accurate computation of dispersion diagrams
- Mode shapes are obtained with no additional computational cost: allows to compute Umov-Poynting vectors
- Results match with the Müller method and experimental results

Conclusion & Outlooks

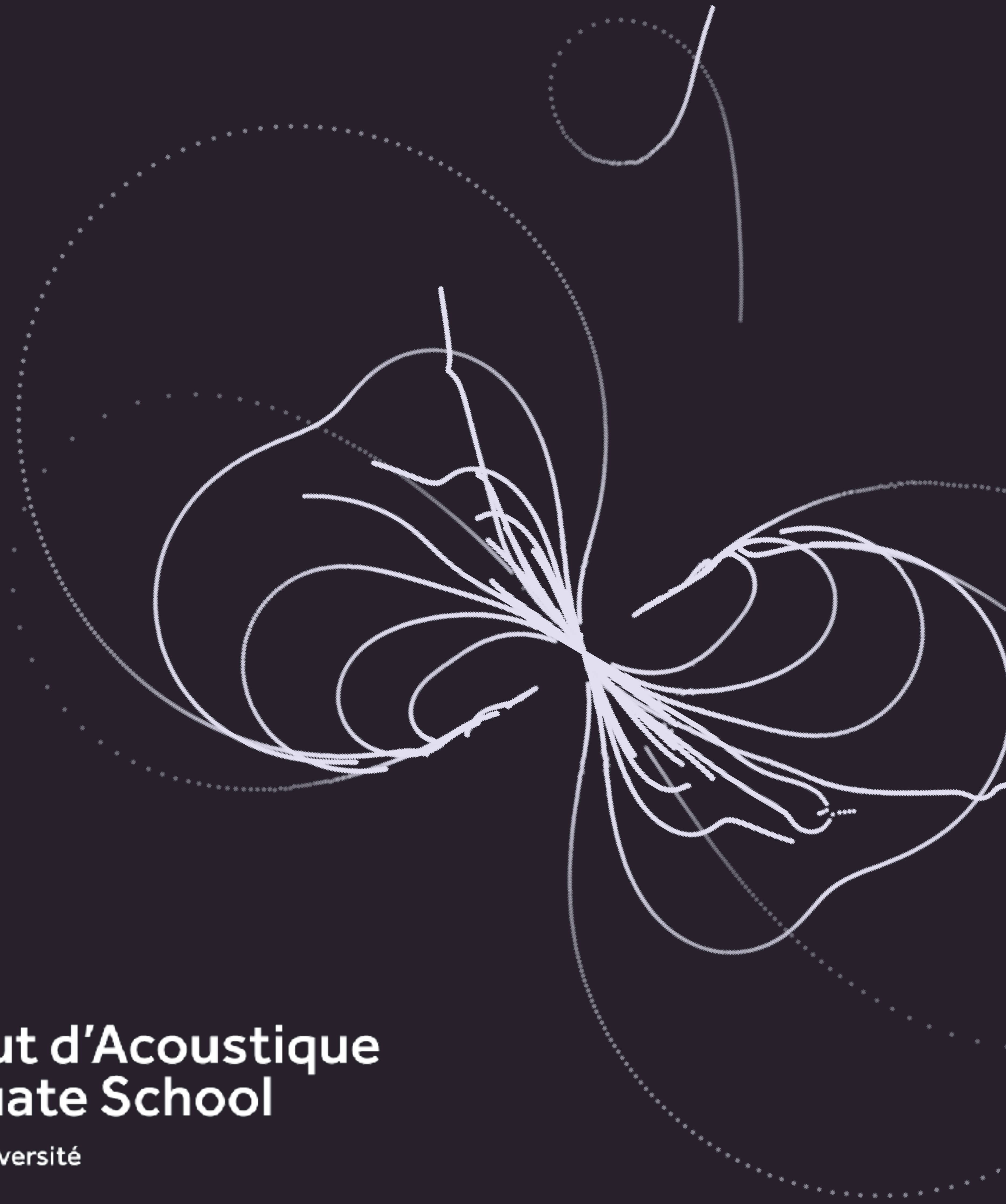


- Numerical method to study guided waves in general multilayer structures
- Fast and accurate computation of dispersion diagrams
- Mode shapes are obtained with no additional computational cost: allows to compute Umov-Poynting vectors
- Results match with the Müller method and experimental results

- Efficiently sort the different branches of the dispersion diagram
  - Study some modes specifically
  - Sorting of the mode shapes
- Add inclusions in the system to provide periodicity: adaptation of the SCM to study periodic structures



Thank you for your attention !



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