Stable model order reduction of vibro-acoustic finite element models with poroelastic materials

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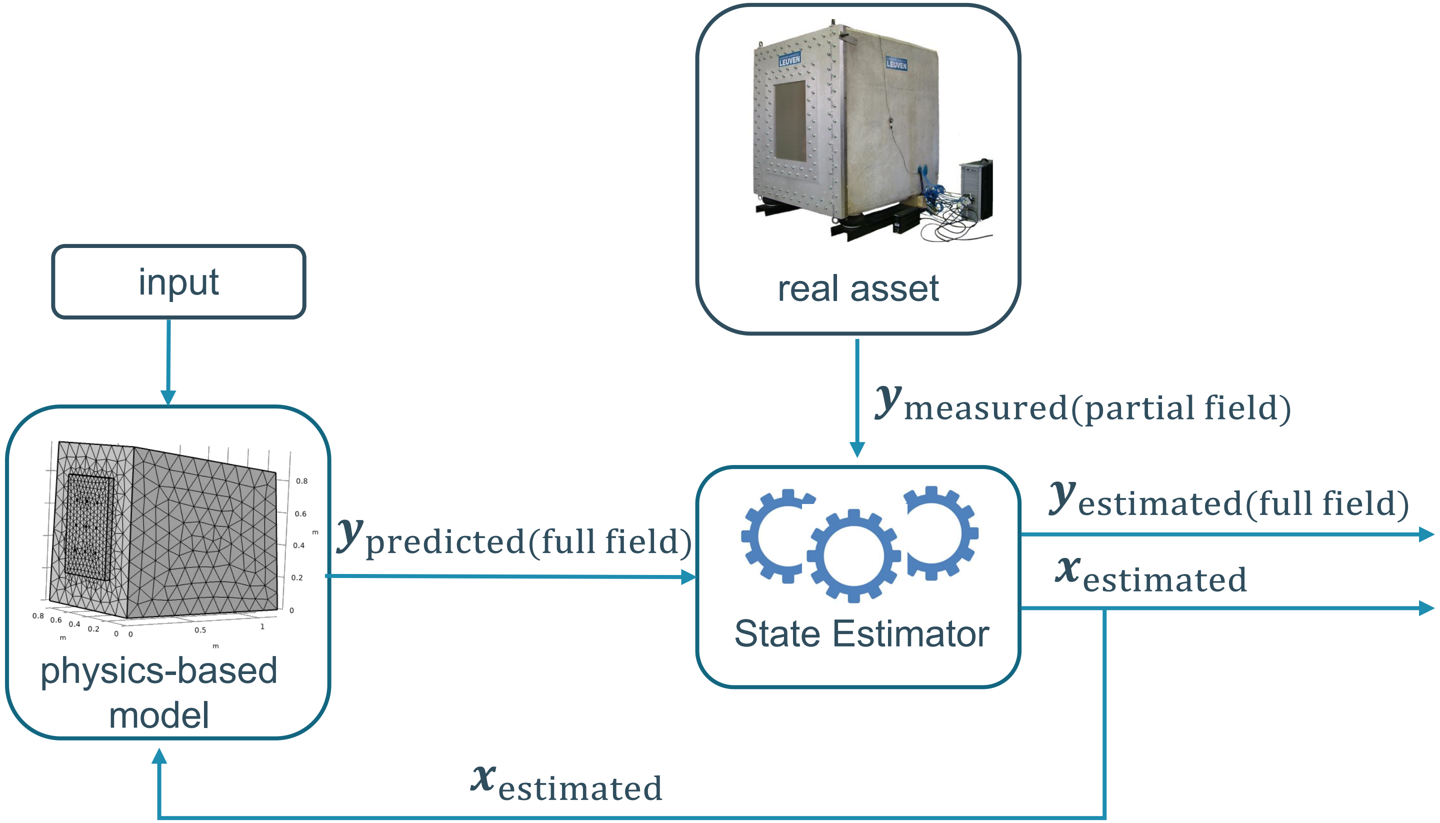
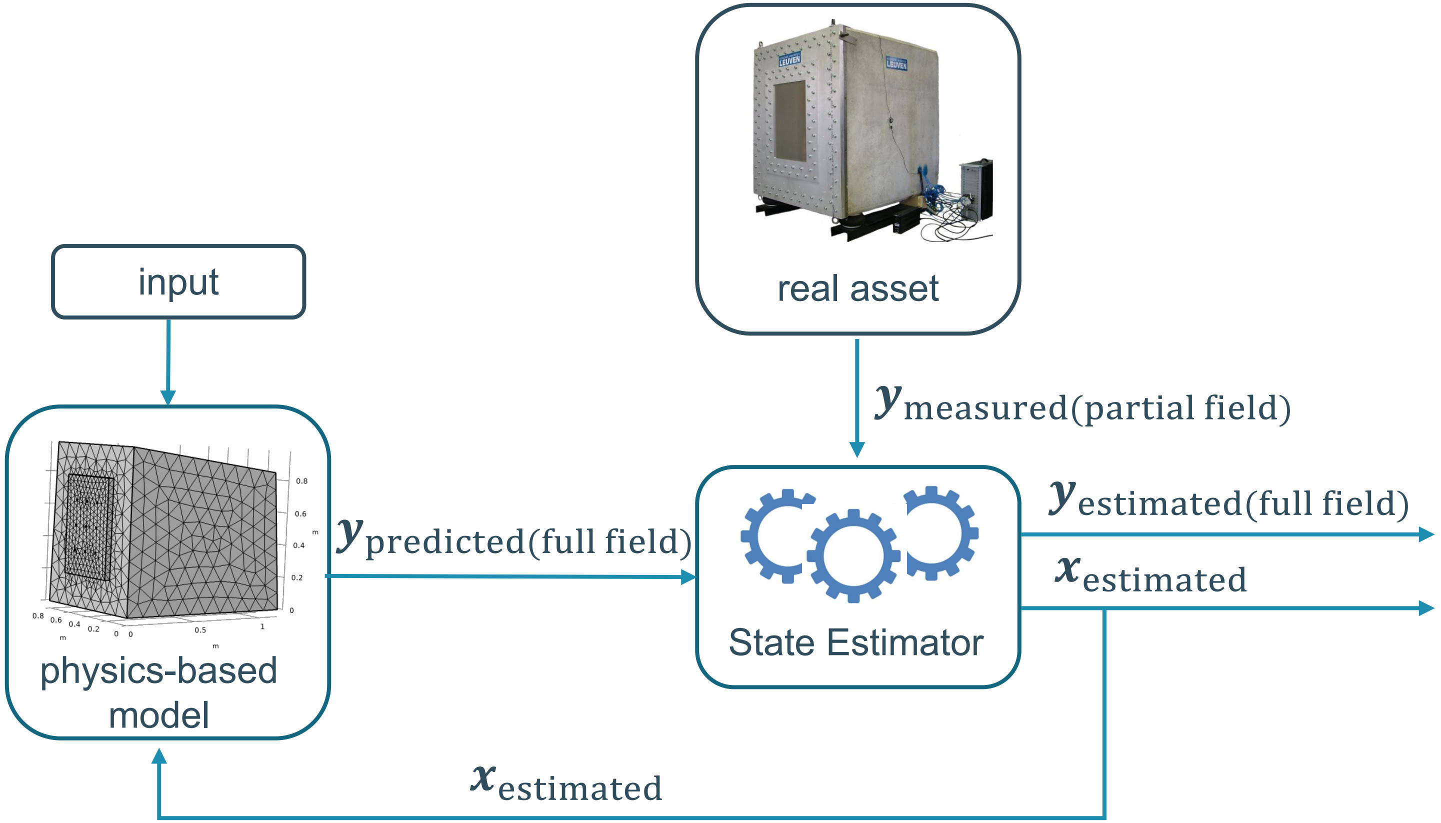
Introduction

- Model based virtual sensing



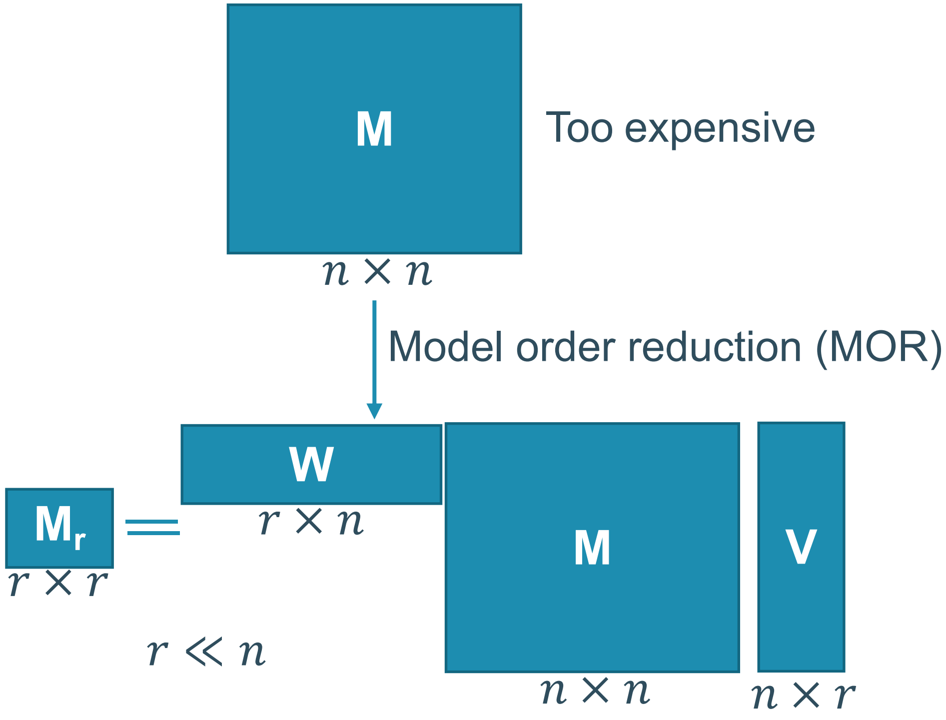
Introduction

- Model based virtual sensing



Physics-based model:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t)$$

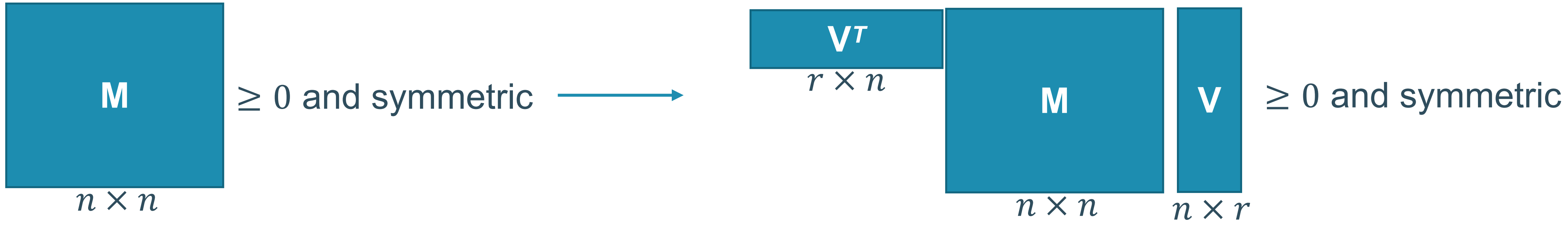


W=V^T→ one-sided MOR

- Stability-preserving MOR?

Stability-preserving conditions[1,2]

- Congruence transform



- Stability-preservation

Lemma

In a second order system if

$$\mathbf{M} = \mathbf{M}^* > \mathbf{0} \text{ and } \mathbf{C} \geq 0, \mathbf{K} = \mathbf{K}^* \geq 0,$$

then this system is stability − preserving under one−sided model order reduction.

[1] Cai Y, van Ophem S, Desmet W, et al. Model order reduction of time-domain vibro-acoustic finite element simulations with admittance boundary conditions for virtual sensing applications[J]. Mechanical Systems and Signal Processing, 2023, 205: 110847.
[2] Cai Y, van Ophem S, Desmet W, et al. Model order reduction of time-domain vibro-acoustic finite element simulations with non-locally reacting absorbers[J]. Computer Methods in Applied Mechanics and Engineering, 2023, 416: 116345.

Biot's equation in u^s-u^t formulation [1,2,3]

$$\nabla\cdot\hat{\boldsymbol{\sigma}}(u^s) = -\omega^2\widetilde{\rho}_s\, u^s - \omega^2\widetilde{\rho}_{eq}\widetilde{\gamma}\, u^t,$$ [1,2]

$$\widetilde{K}_{eq}\,\nabla\nabla\cdot u^t = -\omega^2\widetilde{\rho}_{eq}\widetilde{\gamma}\, u^s - \omega^2\widetilde{\rho}_{eq}\, u^t.$$

↓ FEM

$$\begin{bmatrix} K_s & 0 \\ 0 & \widetilde{K}_{eq}K_0 \end{bmatrix}\begin{bmatrix} u^s \\ u^t \end{bmatrix} + s^2\begin{bmatrix} \tilde{\rho}_s M_0 & \tilde{\gamma}\tilde{\rho}_{eq}M_0 \\ \tilde{\gamma}\tilde{\rho}_{eq}M_0 & \tilde{\rho}_{eq}\mathrm{M}_0 \end{bmatrix}\begin{bmatrix} u^s \\ u^t \end{bmatrix} = \begin{bmatrix} f^s \\ f^t \end{bmatrix}$$

↓ For most porous materials ⟷ Assumption: $\phi \cong 1, K_s \gg K_b, K_s \gg K_f, K_s \gg K_{eq}$

$$\tilde{\gamma}\tilde{\rho}_{eq} = \rho_0 - \tilde{\rho}_{eq}$$
$$\tilde{\rho}_s = \rho_1 + \phi\rho_0 - 2\rho_0 + \tilde{\rho}_{eq}$$

$$s^2\begin{bmatrix} (\rho_1 + \phi\rho_0 - \rho_0)M_0 & 0 \\ 0 & \rho_0 M_0 \end{bmatrix}\begin{bmatrix} u^s \\ u^t \end{bmatrix} + s\underbrace{\frac{\widetilde{K}_{eq}}{s}}_{\text{PR}}\begin{bmatrix} 0 & 0 \\ 0 & \underbrace{\mathbf{K}_0}_{\geq 0} \end{bmatrix}\begin{bmatrix} u^s \\ u^t \end{bmatrix} + s\underbrace{(s(\tilde{\rho}_{eq} - \rho_0))}_{\text{PR}}\underbrace{\begin{bmatrix} M_0 & -M_0 \\ -M_0 & M_0 \end{bmatrix}}_{>0}\begin{bmatrix} u^s \\ u^t \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u^s \\ u^t \end{bmatrix} = \begin{bmatrix} f^s \\ f^t \end{bmatrix}$$

[1] Dazel O, Brouard B, Depollier C, et al. An alternative Biot's displacement formulation for porous materials[J]. The Journal of the Acoustical Society of America, 2007, 121(6): 3509-3516.
[2] Dazel O, Brouard B, Dauchez N, et al. Enhanced Biot's finite element displacement formulation for porous materials and original resolution methods based on normal modes[J]. Acta Acustica united with Acustica, 2009, 95(3): 527-538.
[3] Bécot F X, Jaouen L. An alternative Biot's formulation for dissipative porous media with skeleton deformation[J]. The Journal of the Acoustical Society of America, 2013, 134(6): 4801-4807.

Time-domain porous materials

$$s^2\begin{bmatrix}(\rho_1+\phi\rho_0-\rho_0)M_0 & 0\\ 0 & \rho_0 M_0\end{bmatrix}\begin{bmatrix}u^s\\u^t\end{bmatrix}+s\frac{\widetilde{K}_{eq}}{s}\begin{bmatrix}0 & 0\\0 & \mathbf{K}_0\end{bmatrix}\begin{bmatrix}u^s\\u^t\end{bmatrix}+s(s(\tilde{\rho}_{eq}-\rho_0))\begin{bmatrix}M_0 & -M_0\\-M_0 & M_0\end{bmatrix}\begin{bmatrix}u^s\\u^t\end{bmatrix}+\begin{bmatrix}K_s & 0\\0 & 0\end{bmatrix}\begin{bmatrix}u^s\\u^t\end{bmatrix}=\begin{bmatrix}f^s\\f^t\end{bmatrix}$$

- KYP Lemma

$\frac{\widetilde{K}_{eq}}{s}$: $\begin{matrix}P_1\dot{x}_1=P_1A_1x_1+P_1B_1z_1\\ y_1=C_1x_1+D_1z_1\end{matrix}$ with $\begin{bmatrix}D_1 & C_1\\-P_1B_1 & -P_1A_1\end{bmatrix}\geq 0\ and P_1>0$

$s(\tilde{\rho}_{eq}-\rho_0)$: $\begin{matrix}P_2\dot{x}_2=P_2A_2x_2+P_2B_2z_2\\ y_2=C_2x_2+D_2z_2\end{matrix}$ with $\begin{bmatrix}D_2 & C_2\\-P_2B_2 & -P_2A_2\end{bmatrix}\geq 0\ and P_2>$
0

Time-domain porous materials

$$s^2 \begin{bmatrix} (\rho_1 + \phi\rho_0 - \rho_0)M_0 & 0 \\ 0 & \rho_0 M_0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} + s\frac{\widetilde{K}_{eq}}{s} \begin{bmatrix} 0 & 0 \\ 0 & K_0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} + s(s(\tilde{\rho}_{eq} - \rho_0)) \begin{bmatrix} M_0 & -M_0 \\ -M_0 & M_0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} = \begin{bmatrix} f^s \\ f^t \end{bmatrix}$$

- Cholesky/LDL decomposition

$$K_0 = L_1 L_1^T$$

$$\frac{\widetilde{K}_{eq}}{s} K_0 : \quad \begin{aligned} P_1 \dot{x}_1 &= P_1 A_1 x_1 + P_1 B_1 L_1^T z_1 \\ y_1 &= L_1 C_1 x_1 + K_0 D_1 z_1 \end{aligned}$$

with $\begin{bmatrix} K_0 D_1 & L_1 C_1 \\ -P_1 B_1 L_1^T & -P_1 A_1 \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1 & C_1 \\ -P_1 B_1 & -P_1 A_1 \end{bmatrix} \begin{bmatrix} L_1^T & 0 \\ 0 & I \end{bmatrix} \geq 0$ $and\ P_1 > 0$

$$\begin{bmatrix} M_0 & -M_0 \\ -M_0 & M_0 \end{bmatrix} = \begin{bmatrix} L_{21} & L_{22} \\ L_{23} & L_{24} \end{bmatrix} \begin{bmatrix} L_{21}^T & L_{23}^T \\ L_{22}^T & L_{24}^T \end{bmatrix}$$

$$s(\tilde{\rho}_{eq} - \rho_0)) \begin{bmatrix} M_0 & -M_0 \\ -M_0 & M_0 \end{bmatrix} : \quad \begin{aligned} \begin{bmatrix} P_2 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} &= \begin{bmatrix} P_2 A_2 & 0 \\ 0 & P_2 A_2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} P_2 B_2 L_{21}^T & P_2 B_2 L_{23}^T \\ P_2 B_2 L_{22}^T & P_2 B_2 L_{24}^T \end{bmatrix} \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} \\ y_2 &= \begin{bmatrix} L_{21} C_2 & L_{22} C_2 \\ L_{23} C_2 & L_{24} C_2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} D_2 M_0 & -D_2 M_0 \\ -D_2 M_0 & D_2 M_0 \end{bmatrix} \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} \end{aligned}$$

Time-domain porous materials

$$s^2 \begin{bmatrix} (\rho_1 + \phi\rho_0 - \rho_0)M_0 & 0 \\ 0 & \rho_0 M_0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} + s\frac{\widetilde{K}_{eq}}{s} \begin{bmatrix} 0 & 0 \\ 0 & K_0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} + s(s(\tilde{\rho}_{eq} - \rho_0)) \begin{bmatrix} M_0 & -M_0 \\ -M_0 & M_0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \end{bmatrix} = \begin{bmatrix} f^s \\ f^t \end{bmatrix}$$

↓

$$\underbrace{\begin{bmatrix} (\rho_1 + \phi\rho_0 - \rho_0)M_0 & 0 & 0 & 0 & 0 \\ 0 & \rho_0 M_0 & 0 & 0 & 0 \\ 0 & 0 & P_1 & 0 & 0 \\ 0 & 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & 0 & P_2 \end{bmatrix}}_{M>0} \begin{bmatrix} \ddot{u}^s \\ \ddot{u}^t \\ \dot{x}_1 \\ \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} D_2 M_0 & -D_2 M_0 & 0 & L_{21}C_2 & L_{22}C_2 \\ -D_2 M_0 & K_0 D_1 + D_2 M_0 & L_1 C_1 & L_{23}C_2 & L_{24}C_2 \\ 0 & -P_1 B_1 L_1^{\mathrm{T}} & -P_1 A_1 & 0 & 0 \\ -P_2 B_2 L_{21}^{\mathrm{T}} & -P_2 B_2 L_{23}^{\mathrm{T}} & 0 & -P_2 A_2 & 0 \\ -P_2 B_2 L_{22}^{\mathrm{T}} & -P_2 B_2 L_{24}^{\mathrm{T}} & 0 & 0 & -P_2 A_2 \end{bmatrix}}_{C\geq 0} \begin{bmatrix} \dot{u}^s \\ \dot{u}^t \\ x_1 \\ x_{21} \\ x_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} K_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^s \\ u^t \\ \bar{x}_1 \end{bmatrix} = F$$

Numerical experiments

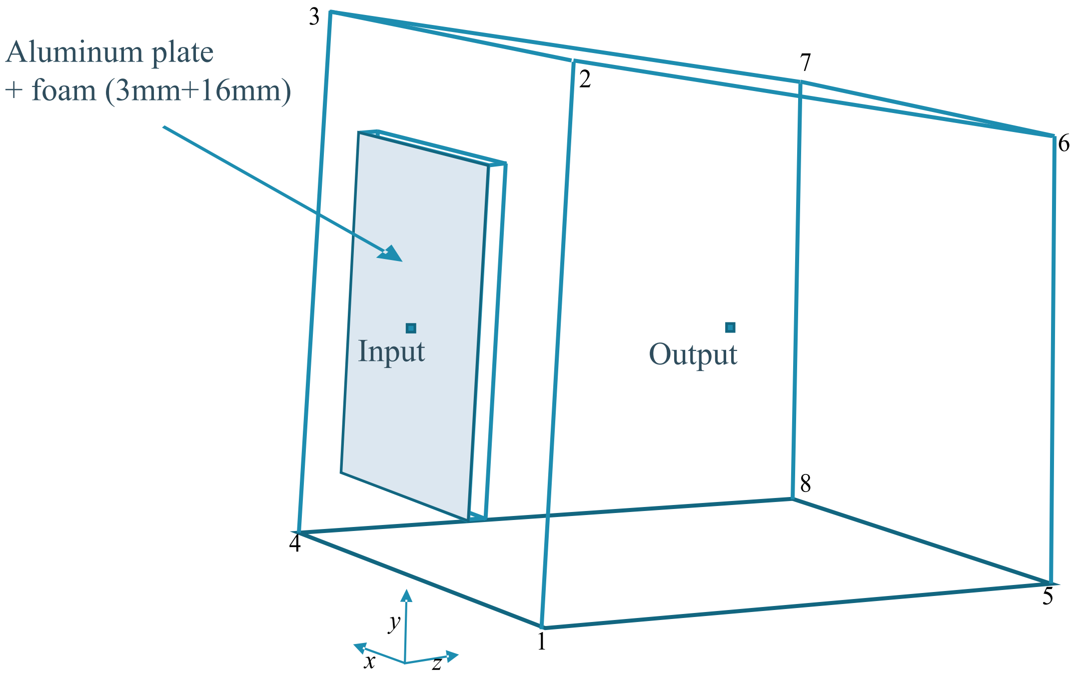


Fig.1. Plate-form-cavity system

Tab. 1. Coordinates of the geometry points

pt.	x/m	y/m	z/m
1	0	0	0
2	0	0.982	0
3	0.778	0.981	0
4	0.815	0	0
5	0	0	1.15
6	0	0.849	1.082
7	0.783	0.848	1.082
8	0.815	0	1.15
in	0.365	0.528	-0.036
out	0.269	0.483	0.573

Tab. 2. Properties of the foam

Tortuosity α_∞	1.98
Density of frame $\rho_1 (kg\ m^{-3})$	16
Flow resistivity $\sigma (N\ m^{-4}\ s)$	65 000
Shear modulus $N\ (kPa)$	18
Poisson coefficient ν	0.3
Porosity ϕ	0.99
Viscous length $\Lambda\ (\mu m)$	37
Thermal length $\Lambda'\ (\mu m)$	121
Length $l\ (mm)$	16

- Foam: JCA model.
- Frequency range: 20-500 Hz.

Numerical experiments

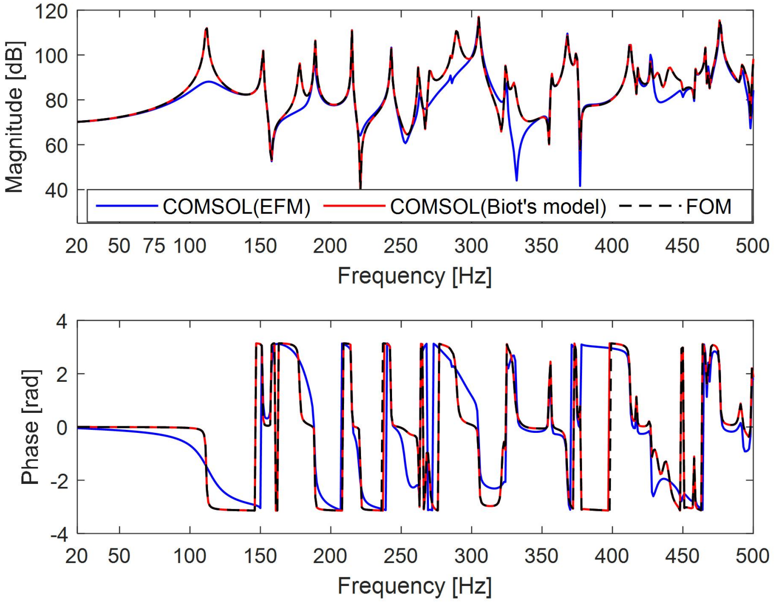


Fig.2. Comparison of the FRFs using the COMSOL software and the proposed method.

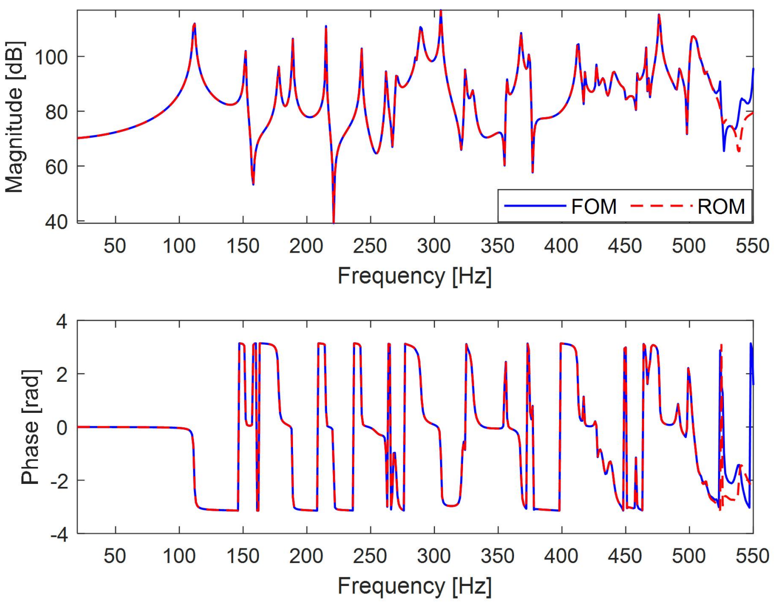


Fig.3. Plate-form-cavity system

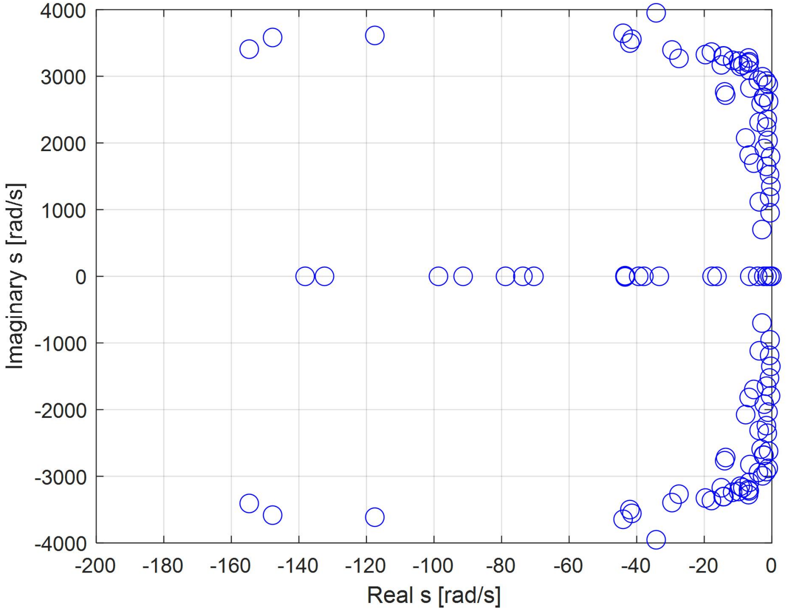


Fig.4. Poles of the ROM.

- COMSOL(Biot's model): Frequency-domain model in u-p formulation.
- DOFs: 42159 (frequency-domain model in u^s-u^t formulation)→96141 (FOM)→137 (ROM).
- Calculate ROM: 540.14s*.

*Personal laptop: a twelve-core 2.6 GHz processor and 32 GB of RAM.

Numerical experiments

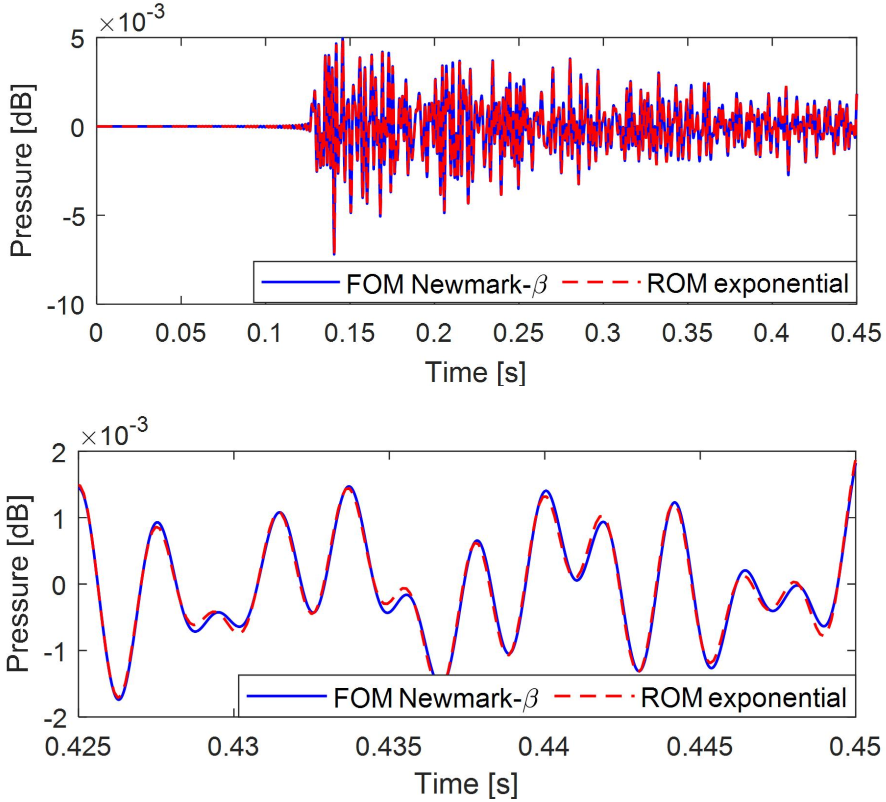


Fig.5. Transient responses of the FOM and the ROM

❑ Enable efficient transient simulations.

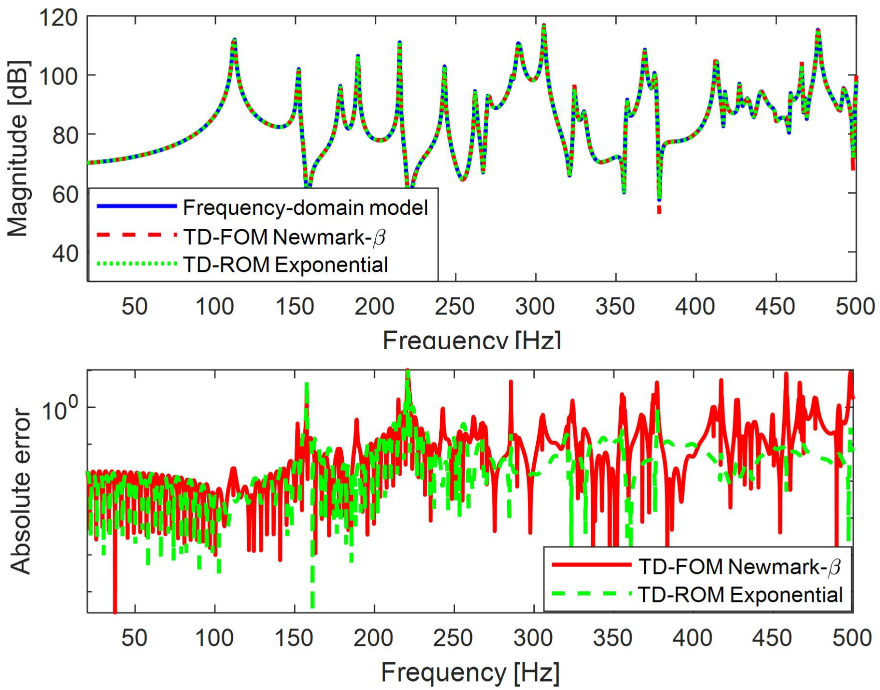


Fig.6. Comparison of the FRFs of the frequency-domain model and the spectrum of the transient response indicated by TD.

Tab.3. Computational time for an input with a duration of 5 s.

Model	Sampling frequency [Hz]	Method	Time [s]
FOM	44100	Newmark-β	48607.3
ROM	8192	exponential	0.53

Conclusions

- A time-domain model of Biot's equation is derived.
- The time-domain model satisfies the stability-preserving conditions.
- The reduced order model enable the efficient time-domain simulations of the vibro-acoustic system with poroelastic materials.

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